Section 2.25. Examples

Note. We now give some of the examples in the text, but present two of them as theorems.

Example 2.25.2. Consider $f(z) = \cosh x \cos y + i \sinh x \sin y$. With $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ we have $u(x,y) = \cosh x \cos y$ and $v(x,y) = \sinh x \sin y$. Recall that $\frac{d}{dx}[\cosh x] = \sinh x$ and $\frac{d}{dx}[\sinh x] = \cosh x$ (see my Calculus 2 [MATH 1920] notes for more properties of the hyperbolic trig functions at 7.8. Hyperbolic Functions). We have that the Cauchy-Riemann equations are satisfied,

$$u_x(x,y) = \sinh x \cos y = v_y \text{ and } u_y(x,y) = -\cosh x \sin y = -v_x$$

for all (x,y). Since the first partial derivatives of u and v exist and are continuous for all (x,y), then by Theorem 2.22.A we have that f is an entire function.

Example 2.25.3. (Theorem 2.25.A.)
Suppose that a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and its conjugate $\overline{f(z)} = u(x,y) - iv(x,y)$ are both analytic in a given domain D. Then f is constant throughout D.

Example 2.25.4. (Theorem 2.25.B.)
Suppose that a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is analytic in a given domain D and that $|f(z)|$ is constant throughout D. Then f is constant throughout D.

Note. We present one more theorem, but leave the proof as Exercise 2.25.7 (Exercise 2.26.7 in the 9th edition of the book).

Exercise 2.25.7. (Theorem 2.25.C.)

Let a function f be analytic everywhere in a domain D. If $f(z)$ is real-valued for all $z \in D$, then $f(z)$ must be constant throughout D.

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