Section 2.26. Harmonic Functions

Note. In this section we define harmonic functions which map $\mathbb{R}^2 \to \mathbb{R}$ and have a number of applications to be explored later. Harmonic functions are then related to analytic functions.

Definition. A real-valued function $H : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* in an open connected set D of the xy-plane if it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

throughout D. The PDE is *Laplace's equation.*

Note. Laplace's equation describes the distribution of heat and electrostatic potential in a two dimensional setting (at equilibrium). Chapters 10 and 11 explore these problems in detail.

Example 2.26.1. Consider $T(x, y) = e^{-y} \sin x$. We have $T_{xx} = -e^{-y} \sin x$ and $T_{yy} = e^{-y} \sin x$, so $T_{xx}(x, y) + T_{yy}(x, y) = 0$. Also, $T(0, y) = 0$, $T(\pi, y) = 0$, $T(x, 0) = \sin x$, and $\lim_{y\to\infty} T(x, y) = 0$. So $T(x, y)$ describes the steady state temperature of a thin plate bounded by the region given in Figure 2.31 where the temperature is held at 0 along the vertical sides and is given by half of a sine wave along the lower boundary. The condition $\lim_{y\to\infty} T(x, y) = 0$ describes a condition on the "boundary at infinity."

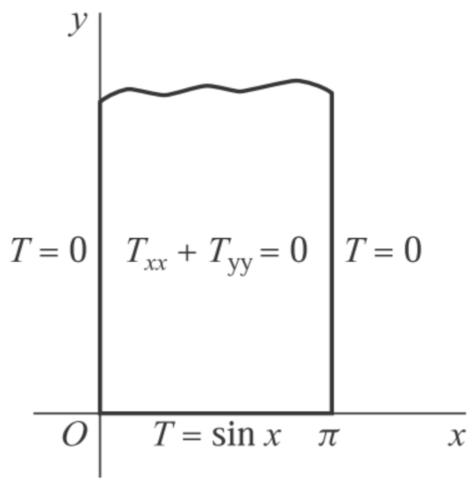

FIGURE 31

Theorem 2.26.1. If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions $u(x,y)$ and $v(x,y)$ are harmonic in D.

Example 2.26.2. The function $f(z) = f(x+iy) = e^{-y}\sin x - ie^{-y}\cos x$ has $u(x,y) = e^{-y}\sin x$ and $v(x,y) = -e^{-y}\cos x$, so that $u_x(x,y) = e^{-y}\cos x = v_y(x,y)$ and $u_y(x,y) = -e^{-y}\sin x = -v_x(x,y)$ for all (x,y). Therefore f satisfies the Cauchy-Riemann equations and the partial derivatives exist and are continuous for all (x,y), so f is an entire function. Therefore by Theorem 2.26.1, both $u(x,y)$ and $v(x,y)$ are harmonic on all of $\mathbb{R}^2$.

Definition. If two given functions $u(x, y)$ and $v(x, y)$ are harmonic in an open connected set in $\mathbb{R}^2$ and their first-order partial derivatives satisfy the Cauchy-Riemann equations throughout the open connected set, then $v(x, y)$ is a *harmonic conjugate* of $u(x, y)$.

Note. We will see soon that $v(x, y)$ may be a harmonic conjugate of $u(x, y)$, but $u(x, y)$ may not be a conjugate of $v(x, y)$.

Theorem 2.26.2. A function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if $v(x, y)$ is a harmonic conjugate of $u(x, y)$.

Example 2.26.4. Suppose that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Then $f(z) = f(x + iy) = u(x, y) + iv(x, y) = z^2$. Since $f(z) = z^2$ is an entire function, then by Theorem 2.26.2 $v(x, y)$ is a conjugate of $u(x, y)$. Now if we consider $g(z) = g(x + iy) = v(x, y) + iu(x, y)$ and try to apply the Cauchy-Riemann equations, we have

$$v_x = 2y \text{ and } u_y = -2y;\ v_y = 2x \text{ and } -u_x = -2x.$$

The first Cauchy-Riemann equation is satisfied only for $y = 0$ and the second Cauchy-Riemann equation is satisfied only for $x = 0$. So the Cauchy-Riemann equations applied to g are satisfied only at 0. That is, g is analytic nowhere. So $u(x, y)$ is not a conjugate of $v(x, y)$ by Theorem 2.26.2.

Note. In Section 9.104, it is shown that every harmonic function $u(x, y)$ has a harmonic conjugate. However, actually finding the harmonic conjugate can be tricky since it involves integration (which can be hard). The next example illustrates a way to find a harmonic conjugate where the integration is easy.

Example 2.26.5. Consider $u(x, y) = y^3 - 3x^2y$. We have $u_x(x, y) = -6xy$, $u_{xx}(x, y) = -6y$, $u_y(x, y) = 3y^2-3x^2$, and $u_{yy}(x, y) = 6y$, so $u_{xx}(x, y)+u_{yy}(x, y) = 0$ and $u(x, y)$ is harmonic in the entire xy-plane. By Theorem 2.26.2, if $v(x, y)$ is a harmonic conjugate of $u(x, y)$ then u and v must satisfy the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$. Here, this implies that $v_y(x, y) = u_x(x, y) = -6xy$. So antidifferentiating with respect to y we have that $v(x, y) = -3xy^2 + \varphi(x)$ where φ is a function of x only. Also $u_y(x, y) = 3y^2-3x^2 = -v_x(x, y) = 3y^2-\varphi'(x)$, so we must have $\varphi'(x) = 3x^2$. Hence $\varphi(x) = x^3 + C$ for some $C \in \mathbb{R}$. So $v(x, y) = -3xy^2 + x^3 + C$. This gives

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C)$$

$$= i\{x^3 - 3xy^2 + i(3x^2y - y^3)\} + iC = iz^3 + iC.$$

Since the components of f have first-order partials which are continuous and satisfy the Cauchy-Riemann equations for all (x, y), then by Theorem 2.22.A f is analytic in the entire complex plane. By Theorem 2.26.2, $v(x, y) = -3xy^2 + x^3 + C$ (where $C \in \mathbb{R}$) is a conjugate of $u(x, y) = y^3 - 3x^2y$.

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