Section 2.27. Uniquely Determined Analytic Functions

Note. In this section we start to see some of the restrictions on analytic functions. We contrast the behavior of analytic functions of a complex variable (using Brown and Churchill's definition) with the behavior of functions of a real variable.

Lemma 2.27.A. Suppose that

(a) a function f is analytic throughout a domain D;

(b) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Then $f(z) = 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

Note. Lemma 2.27.A can be slightly generalized by replacing the condition "$f(z) = 0$" with the condition "$f(z) = c$" for some constant $c \in \mathbb{C}$ and by replacing the conclusion that "$f(z)$ is identically equal to zero" with the conclusion "$f(z) = c$." The proof of the generalization follows by applying Lemma 2.27.A to the function $f(z) - c$.

Note. Lemma 2.27.A can be generalized to state that if two analytic functions on domain D, f and g, are equal on a set that has a limit point in D, then $f(z) = g(z)$ for all $z \in D$. Of course a subdomain of D or a line segment in D contains a limit point in D. See Corollary IV.3.3 in my online notes for Complex Analysis 1 [MATH 5510] on IV.3. Zeros of Analytic Functions.

Theorem 2.27.A. A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.

Example. Theorem 2.27.A does not apply to functions of a real variable. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \\ e^{-1/x^2} & \text{if } x \in (0, \infty). \end{cases}$$

notice that f is clearly differentiable for all real x, except possibly for $x = 0$. In fact, $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$ (this can be confirmed by considering the derivative at 0 in terms of one-sided limits). So f is differentiable on all of $\mathbb{R}$. If we were to try to use Brown and Churchill's definition of "analytic" in the real setting, then we get that a function of a real variable is "analytic" on a set if it is differentiable in a neighborhood of each point of the set. If this is the case (which it is not; the definition of "analytic" in the real setting involves the existence of a power series), then we see that f as defined in this example is analytic. However, its behavior is not uniquely determined on a line segment, say $(-\infty, 0)$ here, since the function $g(x) \equiv 0$ satisfies $f(x) = g(x)$ for all $x \in (-\infty, 0)$, but f and g are different functions of a real variable. By the way, function f given here is the standard example of an infinitely differentiable function of a real variable at $a = 0$ which does not have a power series representation of the form $f(x) = \sum_{n=0}^{\infty} a_k(x-a)^n$. For details, see my online notes for Analysis 2 (MATH 4227/5227) on 8.3. Taylor Series.

Note. Let D_1 and D_2 be domains in $\mathbb{C}$ with f_1 analytic on D_1 and f_2 analytic on D_2. If for $z \in D_1 \cap D_2 \neq \varnothing$ we have $f_1(z) = f_2(z)$, then f_1 can be extended from D_1 to $D_1 \cup D_2$. The function $f_2(z)$ is called the *analytic continuation* of f_1 into D_2. However, an analytic continuation does not always exist. When an analytic continuation of f_1 exists, it is unique by Theorem 2.27.A. However, we can have a situation where f_2 is an analytic continuation of f_1 into D_2 and f_3 is an analytic continuation of f_2 into a domain D_3 which intersects D_1 (see Figure 34), but $f_3(z)$ is not an analytic continuation of $f_1(z)$ into D_3. This is illustrated in Exercise 2.28.2 (Exercise 2.29.2 in the 9th edition of the book). Analytic continuation is explored in Chapter IX of Conway's *Functions of One Complex Variable I* (see my Additional Class Notes for Complex Analysis 2 [MATH 5520]).

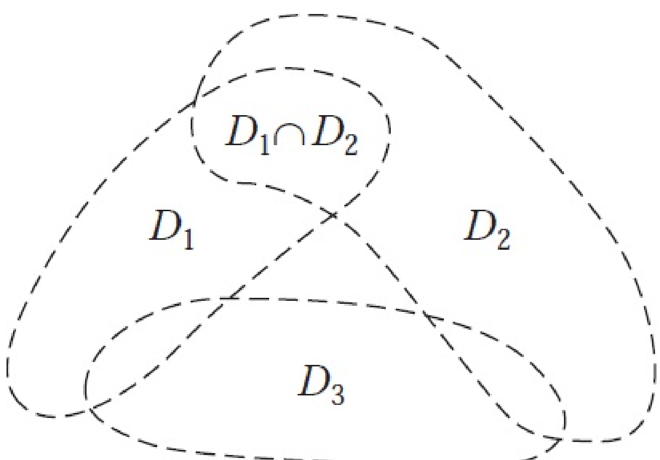

FIGURE 34

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