Section 2.28. Reflection Principle

Note. In this section we state and prove a single result. It is applicable to many familiar functions from the real setting.

Theorem 2.28.A. (Reflection Principle) Suppose that a function f is analytic in some domain D which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to that axis. Then $\overline{f(z)} = f(\overline{z})$ for each point z in the domain if and only if $f(x)$ is real for each point x on that segment.

Note. Notice that any polynomial with real coefficients, $p(z) = \sum_{k=0}^{n} a_k z^k$ where each $a_k \in \mathbb{R}$, we have that p is an entire function and $p(x)$ is real for all $x \in \mathbb{R}$. So Theorem 2.28.A implies that $\overline{p(z)} = p(\overline{z})$ (of course this also follows by properties of conjugates as well). Once we encounter more functions in the next chapter, we will see many more functions satisfying this property that real inputs give real outputs; namely, the functions e^z, $\sin z$, and $\cos z$. The other trig functions and hyperbolic trig functions also fall into this category.

Example. Some functions which are *not* real for real input are $f_1(z) = iz$ and $f_2(z) = z + i$. So these two functions do not satisfy the Reflection Principle.

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