Chapter 3. Elementary Functions

Section 3.29. The Exponential Function

Note. In this section we explore the function e^z in a little more detail.

Note. In section 2.14, we defined $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$ (based on Euler's formula from Section 1.6).

Lemma 3.29.A. For all $z_1, z_2 \in \mathbb{C}$ we have $e^{z_1}e^{z_2} = e^{z_1+z_2}$.

Corollary 3.29.A. For all $z_1, z_2 \in \mathbb{C}$ we have $e^{z_1}/e^{z_2} = e^{z_1-z_2}$. Also, $1/e^z = (e^z)^{-1} = e^{-z}$.

Note. We know from Example 2.22.1 that $\dfrac{d}{dz}[e^z] = e^z$ and so e^z is an entire function. Since

$$|e^z| = |e^{x+iy}| = |e^x e^{iy}| = |e^x||\cos y + i \sin y| = \left| e^x \sqrt{\cos^2 y + \sin^2 y} \right| = |e^x| = e^x,$$

then $e^z \neq 0$ for all $z \in \mathbb{C}$. Also $\arg(e^z) = \arg(e^{x+iy}) = \{y + 2n\pi \mid n \in \mathbb{Z}\}$. A surprising result is that e^z is a periodic function. Notice that $e^{z+2\pi i} = e^z$ so that the period of e^z is $2\pi i$.

Note 3.29.A. We also have for any $z = re^{i\theta} \in \mathbb{C}$ where $z \neq 0$, that

$$e^{\ln(|z|)+i\ \arg z} = e^{\ln|z|}e^{i\ \arg z} = |z|e^{i\theta} = re^{i\theta} = z.$$

So the range of e^z is $\mathbb{C} \setminus \{0\}$. This, combined with the periodic nature of e^z will play a role in defining the inverse of e^z (that is, defining the complex logarithm function) in the next two sections.

Example. Find all $z = x + iy$ such that $e^z = 1 + i$.

Solution. By Note 2.29.A we have $e^{\ln|w|+i\arg(w)} = w$. Here with $w = 1 + i$ we have $\ln|w| = \ln|1+i| = \ln\sqrt{2} = (1/2)\ln 2$ and $\arg(w) = \arg(1+i) = (\pi/4+2n\pi)i$ where $n \in \mathbb{Z}$. So to solve $e^z = 1+i$ we take $z = \ln|w| + i\arg(w) = \ln|1+i| + i\arg(1+i) = \frac{1}{2}\ln 2 + i(2n + 1/4)\pi$ where $n \in \mathbb{Z}$. $\square$

Revised: 4/2/2024