# **Section 3.30.** The Logarithm Function

**Note.** In this section we begin defining a complex logarithm function. In so doing, we will define inverses of various restricted versions of $e^z$. This will provide some challenges since $e^z$ is a periodic function with period $2\pi i$. The challenges are resolved with the definition of the principal branch of the logarithm.

**Note.** At the end of Section 3.29 we saw that for $z \neq 0$, $z = e^{\ln(|z|)+i\ \arg(z)}$. The set $\arg(z)$ consists of an infinite number of values, any two of which differ by an integer multiple of $2\pi i$. This leads to the following definition.

**Definition.** For any $z \in \mathbb{C}$ with $z \neq 0$, define the "multiple-valued" function

$$\log z = \ln |z| + i\ \arg(z).$$

**Note 3.30.A.** Arguably, there is no such thing as a "multiple-valued" function by the very definition of "function." Like the set valued "$\arg(z)$" encountered in Section 1.8, we can similarly think of $\log z$ as defined here as set valued. Brown and Churchill approach it slightly differently. They take the unique value in $\arg(z)$ in the interval $(-\pi, \pi]$ and denote it as $\Theta$. They then define

$$\log z = \ln |z| + i(\Theta + 2n\pi) \text{ where } n \in \mathbb{Z}.$$

This is equivalent to the definition above. Sometimes the value $\Theta$ is called the *principal argument* of $z$ (see Section 1.6. Exponential Form). We'll use it to define the principal branch of the logarithm. The interval $(-\pi, \pi]$ is not universally accepted for the principal value. Notice that we have $e^{\log z} = z$ for all $z \neq 0$.

**Example.** In Section 3.29 we saw that if $e^z = 1 + i$ then $z = x + iy = \ln(\sqrt{2}) + (\pi/4 + 2n\pi)i$ for $n \in \mathbb{Z}$. Therefore $\log(1 + i) = \ln(\sqrt{2}) + (\pi/4 + 2n\pi)i$ for $n \in \mathbb{Z}$ or, more appropriately, $\log(1 + i) = \{\ln(\sqrt{2}) + (\pi/4 + 2n\pi)i \mid n \in \mathbb{Z}\}$.

**Definition.** The *principal value of* $\log z$ is based on the value of $\arg(z)$ in $(-\pi, \pi]$ which we denote $\Theta$. We define the principal value of $\log z$ as

$$\text{Log } z = \ln|z| + i\Theta.$$

**Note.** We have that Log $z$ is a "single-valued" function (that is, an actual function). The relationship between Log $z$ and $\log z$ is

$$\log z = \text{Log } z + 2n\pi i \text{ for } n \in \mathbb{Z}.$$

(More appropriately, $\log z = \{\text{Log } z + 2n\pi i \mid n \in \mathbb{Z}\}$.)

**Example 3.30.2.** With $z = 1$ we have $|z| = 1$ and $\arg(z) = \{2n\pi \mid n \in \mathbb{Z}\}$. Therefore $\log 1 = \ln 1 + 2n\pi i = 2n\pi i$ for $n \in \mathbb{Z}$. (More appropriately, $\log 1 = \{2n\pi i \mid n \in \mathbb{Z}\}$.) But Log $1 = 0$.

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