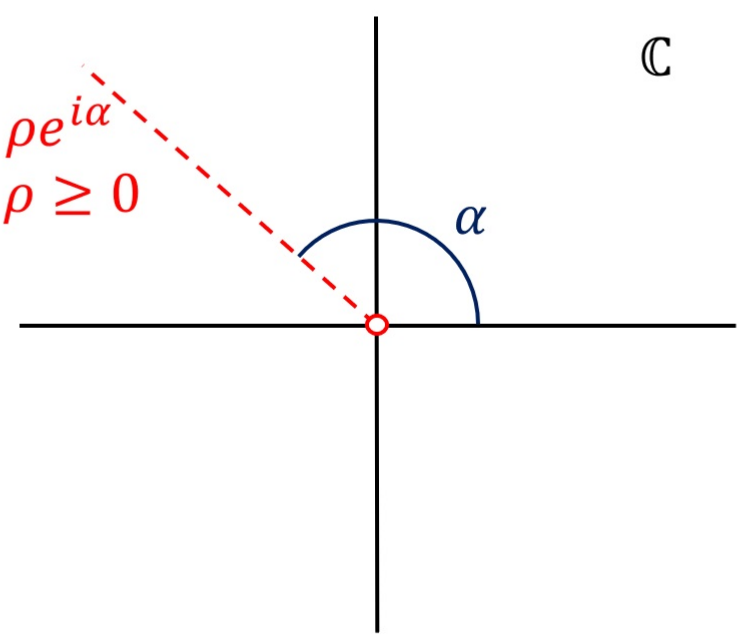
# **Section 3.31.** Branches and Derivatives of Logarithms

**Note.** In this section Brown and Churchill continue to resolve the "many-valued" versus "single-valued" functions situation with their definition of $\log z$ and $\mathrm{Log}\, z$.

**Note 3.31.A.** Let $\alpha$ denote any real number. For any $z \in \mathbb{C}$ such that $z = re^{i\theta}$ where $r > 0$ and $\theta \in (\alpha, \alpha+2\pi)$, we could define $\log z = \log re^{i\theta} = \ln r + i\theta$. In polar coordinates we have $\log z = u(r,\theta) + iv(r,\theta)$ where $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$. Throughout the domain $D = \mathbb{C} \setminus \{z = \rho e^{i\alpha} \mid \rho \geq 0\}$ (see the graph below) we have that $\log z$ is defined and satisfies the polar form of the Cauchy-Riemann equations: $ru_r(r,\theta) = 1 = v_\theta(r,\theta)$ and $u_\theta(r,\theta) = 0 = -rv_r(r,\theta)$. So by Theorem 2.23.A, $\log z$ (so defined) is analytic in $D$ and $\dfrac{d}{dz}[\log z] = e^{-i\theta}(u_r(r,\theta) + iv_r(r,\theta)) = e^{-i\theta}(1/r + i0) = 1/(re^{i\theta}) = 1/z$. As a special case, with the principal branch of the logarithm we have with $\alpha = -\pi$ that $\dfrac{d}{dz}[\mathrm{Log}\ z] = 1/z$ for all $z \in \mathbb{C} \setminus \{z \in \mathbb{C} \mid z \in \mathbb{R}, z \leq 0\}$.



**Note.** The function $\log z$ defined in the previous note (unfortunately with the same notation as used for the multiple-valued function "$\log z$" from the previous section) is called a *branch* of the logarithm and the set $\{z = \rho e^{i\alpha} \mid \rho \geq 0\}$ on which $\log z$ is not defined is the *branch cut*. The point $z = 0$ is the *branch point*. These ideas will carry over to other multiple-valued functions in this chapter, each of which will result from trying to take the inverse a periodic function.

*Revised: 3/27/2020*