Section 3.32. Some Identities Involving Logarithms

Note. In this section we state and prove a few properties of the logarithm.

Lemma 3.32.A. For the multiple-valued "function" $\log z$ defined in Section 3.30, we have for all nonzero $z_1, z_2 \in \mathbb{C}$ that

$$\log(z_1 z_2) = \log z_1 + \log z_2.$$

Note. The principal value of the logarithm, $\operatorname{Log} z$, need not, in general, satisfy the property given in Lemma 3.32.A. However, as is to be shown in Exercise 3.32.1 (and Example 3.34.2 in the 9th edition of the book), if $\operatorname{Re}(z_1) > 0$ and $\operatorname{Re}(z_2) > 0$, then $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$.

Lemma 3.32.B. For the multiple-valued "function" $\log z$ defined in Section 3.30, we have for all nonzero $z_1, z_2 \in \mathbb{C}$ that

$$\log(z_1/z_2) = \log z_1 - \log z_2.$$

Proof. The proof is to be given in Exercise 3.32.3 (in Exercise 3.34.2 in the 9th edition of the book). □

Lemma 3.32.C. For any nonzero $z \in \mathbb{C}$, for all $n \in \mathbb{Z}$ we have $z^n = e^{n \log z}$.

Lemma 3.32.D. For any nonzero $z \in \mathbb{C}$, we have that for $n = 1, 2, 3, \ldots$

$$\exp\left(\frac{1}{n}\log z\right)$$

is a set consisting of n distinct elements each of which is an nth root of z (that is, when raised to the nth power gives z).

Note. In Exercise 3.32.5 (in Exercise 3.34.4 in 9th edition) it is shown that Lemma 3.32.D also holds for $n \in \mathbb{Z}$.

Note. We will apply these lemmas in the next section.

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