Section 3.33. Complex Exponents

Note. In this section we deal with raising a complex number to a complex power. This will be based on logarithms and branches of logarithms and so will lead to the multiple-valued thing again (and the idea of principal values which resolve this issue).

Definition. When $z \neq 0$, we define the "multiple-valued" function z^c for $c \in \mathbb{C}$ as $z^c = e^{c \log z}$.

Note. Since $\log z$ is multiple-valued, then we expect z^c to be multiple valued. The definition is similar to that of the function a^x in terms of e^x in the real setting (see my online notes for Calculus 2 on 7.4. a^x and $\log_a x$; these notes are not based on an "early transcendentals" calculus text and so the theoretical development of logarithms and exponentials is more rigorous). Also notice that this definition of z^c is consistent with the case $c = n \in \mathbb{N}$ as given in Lemma 3.32.C.

Example 3.33.1. Calculate i^{-2i}. We have $i^{-2i} = \exp(-2i \log i)$ where

$$\log i = \ln |i| + i \arg(i) = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(\frac{\pi}{2} + 2n\pi\right),\ n \in \mathbb{Z}.$$

So the multiple-valued result is

$$i^{-2i} = \exp(-2i[i(\pi/2 + 2n\pi)]) = \exp(\pi + 4n\pi) = \exp(\pi(1 + 4n))$$

for $n \in \mathbb{Z}$. Notice that all of the values of i^{-2i} are real and distinct (since the real exponential function is one to one).

Definition. Let $\log z$ represent some branch of the logarithm. That is, $\log z = \ln|z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$. Then a *branch* of z^c is given by $z^c = e^{c\log z}$. The *principal branch* of z^c is based on the principal branch of the logarithm (for which we take $\arg(z) \in (-\pi, \pi)$): $z^c = e^{c\,\mathrm{Log}\, z}$. The principal branch of z^c gives *principal values* of z^c, which Brown and Churchill denote "P.V. z^c." (Notice that the principal branch of z^c is not defined for nonpositive real numbers).

Note. A common misconception is that the square root function in the real setting is "2-valued." It is easy to trick a freshman level math student into thinking that $\sqrt{9}$ is ± 3. Of course this is *not* the case and $\sqrt{9} = 3$. This is because $\sqrt{x}$ is a *function* (just ask a calculator what $\sqrt{9}$ is). If you want both the positive and negative square roots (which may well be the case in an application) then you must "ask" for both the positive and negative square roots: $\pm\sqrt{9} = \pm 3$. This is related to our situation with branches of z^c. With $c = 1/2$, we have that the principal branch of $z^{1/2}$ for $z = 9$ gives

$$\text{P.V. } 9^{1/2} = e^{(1/2)\mathrm{Log}\, 9} = e^{(1/2)(\ln 9 + i0)} = e^{(1/2)\ln 9} = \left(e^{\ln 9}\right)^{1/2} = 9^{1/2} = \sqrt{9} = 3.$$

Theorem 3.33.A. For any branch of z^c, we have $\dfrac{d}{dz}[z^c] = cz^{c-1}$ where the branch of z^{c-1} is based on the same branch of the logarithm on which z^c is based.

Example. The principal value of i^i is

$$i^i = \exp(i\,\mathrm{Log}\, i) = \exp(i[\ln|i| + i\pi/2]) = \exp(-\pi/2),$$

since $\mathrm{Log}\, i = \pi/2$. Again, notice that the principal value of i^i is real.

Example 3.33.3. The principal branch of $z^{2/3}$ is

$$\exp((2/3)\text{Log } z) = \exp((2/3)\ln|z| + (2/3)i\Theta) = \sqrt[3]{|z|^2}\exp(2\Theta i/3)$$

$$= \sqrt[3]{|z|^2}(\cos 2\Theta/3 + i\sin 2\Theta/3)$$

where Θ is the principal argument of z (notice that we must have $\Theta \neq \pi$ and so z cannot be a nonpositive real number when using the principal branch).

Note. The definition of z^c implies that the (multiple-valued) *exponential function with base c* is $c^z = e^{z\log c}$. Branches and the principal branch of c^z is similarly defined. We find:

$$\frac{d}{dz}[c^z] = c^z \log c.$$

In practice you are unlikely to use any exponential function other than the natural exponential function, e^z.

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