Section 3.34. Trigonometric Functions

Note. In this section we define the complex trigonometric functions and some of their properties. We also relate them to the real hyperbolic trig functions.

Note. To recap, we have defined the complex exponential function in terms of the real exponentiation function, the real sine function, and the real cosine function: $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. We continue this trend of defining new functions in terms of existing ones.

Note. Since Euler's formula gives (for $x \in \mathbb{R}$) $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$, then we can solve for $\cos x$ and $\sin x$ to get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This motivates the following definition.

Definition. For all $z \in \mathbb{C}$ define the *cosine* and *sine* functions as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Note 3.34.A. Since e^z is an entire function, then $\cos z$ and $\sin z$ are entire functions. Notice that the cosine and sine functions of a complex variable agree with the cosine and sine functions of a real variable when z is real. Therefore, by Theorem 2.27.A, $\cos z$ and $\sin z$ are the unique entire functions which agree with $\cos x$

and $\sin x$. They satisfy the expected differentiation properties:

$$\frac{d}{dx}[\cos z] = \frac{d}{dz}\left[\frac{e^{iz}+e^{-iz}}{2}\right] = \frac{ie^{iz}-ie^{-iz}}{2} = \frac{-e^{iz}+e^{-iz}}{2i} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z$$

$$\frac{d}{dx}[\sin z] = \frac{d}{dz}\left[\frac{e^{iz}-e^{-iz}}{2i}\right] = \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z.$$

From the definitions we also immediately have $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, and $e^{iz} = \cos z + i\sin z$.

Note 3.34.B. In Exercises 3.34.2 and 3.34.3 (Exercises 3.38.2 and 3.38.3 in the 9th edition of the book) you are asked to verify the summation formulas

$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \text{ and } \sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

These identities lead to the double angle formulas

$$\sin 2z = 2\sin z\cos z \text{ and } \cos 2z = \cos^2 z - \sin^2 z$$

and the cofunction relations

$$\cos(z+\pi/2) = \sin z \text{ and } \sin(z+\pi/2) = \cos z$$

(sine and cosine are COfunctions because they give equal values for COmplementary "angles"). We also have the trigonometric version of the Pythagorean Theorem: $\cos^2 z + \sin^2 z = 1$. The summation formulas give the periodic behavior of sine and cosine:

$$\cos(z+2\pi) = \cos z \text{ and } \sin(z+2\pi) = \sin z$$

and the following phase shift properties:

$$\cos(z+\pi) = -\cos z \text{ and } \sin(z+\pi) = -\sin z.$$

Note. Recall the hyperbolic cosine and sine functions for a real variable y:

$$\cosh y = \frac{e^y + e^{-y}}{2} \text{ and } \sinh y = \frac{e^y - e^{-y}}{2}.$$

These are pronounced "cosh" and "sinh," respectively. For properties of the hyperbolic trig functions, see my online Calculus 2 (MATH 1920) notes on 7.3. Hyperbolic Functions.

Note. We can use imaginary numbers to relate cosine and cosh, and sine and sinh as:

$$\cos iy = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y,$$
$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i \sinh y.$$

Lemma 3.34.A. The real and imaginary parts of $\cos z$ and $\sin z$ can be expressed in terms of $\sin x$, $\cos x$, $\sinh y$, and $\cosh y$, where $z = x + iy$, as:

$$\sin z = \sin x \cosh y + i \cos x \sinh y \text{ and } \cos z = \cos x \cosh y - i \sin x \sinh y.$$

Note 3.34.C. By Lemma 3.34.A and the facts that $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$ we have that (this is Exercise 3.34.7; it is Exercise 3.38.7 in the 9th edition of the book):

$$\begin{aligned} |\sin z|^2 &= (\sin x \cosh y)^2 + (\cos x \sinh y)^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y = \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y \\ &= \sin^2 x + \sinh^2 y, \end{aligned}$$

and

$$\begin{aligned}
|\cos z|^2 &= (\cos x \cosh y)^2 + (\sin x \sinh y)^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y = \cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
&= \cos^2 x + \sinh^2 y.
\end{aligned}$$

Since $\sinh y = (e^y - e^{-y})/2$ is unbounded (consider $y \to \infty$) then neither $\cos z$ nor $\sin z$ is bounded (they grow exponentially along the positive imaginary axis), unlike their real counterparts both of which are bounded below by -1 and above by 1. This turns out not to be surprising since we will see in Liouville's Theorem in Section 4.53 that the only bounded entire functions are constant functions.

Definition. A *zero* of a function $f(z)$ is a $z_0 \in \mathbb{C}$ for which $f(z_0) = 0$.

Note. The zeros of a polynomial function are often called "roots" (especially in an algebraic setting). The study of the location of the zeros of functions of a complex variable is a vibrant area of contemporary study. We know the zeros of the real functions $\cos x$ and $\sin x$. The following result proves that these are the only zeros for $\cos z$ and $\sin z$, respectively.

Lemma 3.34.B. The only zeros of $\sin z$ are the real numbers $z = n\pi$ where $n \in \mathbb{Z}$. The only zeros of $\cos z$ are the real numbers $z = \pi/2 + n\pi$ where $n \in \mathbb{Z}$.

Definition. We now define the four remaining trigonometric functions as functions of a complex variable. We have

$$\tan z = \frac{\sin z}{\cos z},\ \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z},\ \csc z = \frac{1}{\sin z}.$$

Note. Notice that none of $\tan z$, $\cot z$, $\sec z$, nor $\csc z$ are entire functions. In fact, they have singularities at the zeros of either $\sin z$ (for $\cot z$ and $\csc z$) or $\cos z$ (for $\tan z$ and $\sec z$). The summation formulas for $\cos z$ and $\sin z$ allow us to establish the periodic nature of these functions (the periods of $\tan z$ and $\cot z$ are π and the periods of $\sec z$ and $\csc z$ are 2π). We can use the definitions in terms of $\cos z$ and $\sin z$ to establish the familiar differentiation properties:

$$\frac{d}{dz}[\tan z] = \sec^2 z,\ \frac{d}{dz}[\cot z] = -\csc^2 z,$$
$$\frac{d}{dz}[\sec z] = \sec z \tan z,\ \frac{d}{dz}[\csc z] = -\csc z \cot z.$$

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