Section 3.35. Hyperbolic Functions

Note. In this section we define the six hyperbolic trigonometric functions and state some identities and properties.

Note. Recall the definition of the real hyperbolic trig functions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}.$$

Just as in the last section, we define new functions of a complex variable in terms of previously constructed functions.

Definition. For any $z \in \mathbb{C}$ define the hyperbolic cosine and hyperbolic sine as:

$$\cosh z = \frac{e^z + e^{-z}}{2} \text{ and } \sinh z = \frac{e^z - e^{-z}}{2}.$$

Note 3.35.A. Since e^z is an entire function, then $\cosh z$ and $\sinh z$ are entire functions. Notice that the hyperbolic cosine and sine functions of a complex variable agree with the hyperbolic cosine and sine functions of a real variable when z is real. Therefore, by Theorem 2.27.A, $\cosh z$ and $\sinh z$ are the unique entire functions which agree with $\cosh x$ and $\sinh x$. They satisfy the expected differentiation properties (see Exercise 3.35.a; see Exercise 3.39.1 in the 9th edition of the book):

$$\frac{d}{dx}[\cosh z] = \sinh z \text{ and } \frac{d}{dz}[\sinh z] = \cosh z.$$

Note 3.35.B. We can use the definitions of the trigonometric functions in terms of exponential functions to deduce the following (see Exercise 3.39.A):

$$-i\sinh(iz) = \sin z,\ \cosh(iz) = \cos z,$$

$$-i\sin(iz) = \sinh z,\ \cos(iz) = \cosh z.$$

The following are confirmed in the exercises:

$$\sinh(-z) = -\sinh z,\ \cosh(-z) = \cosh z,\ \cosh^2 z - \sinh^2 z = 1,$$

$$\sinh(z_1 + z_2) = \sinh z_2 \cosh z_2 + \cosh z_1 \sinh z_2,$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2,$$

$$\sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$$

$$\cosh z = \cosh(z + iy) = \cosh x \cos y + i \sinh x \sin y,$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y,\ |\cosh z|^2 = \sinh^2 x + \cos^2 y.$$

Note 3.35.C. Since $\cos z$ and $\sin z$ have period 2π, it follows from the identities $\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$ that $\cosh z$ and $\sinh z$ have period $2\pi i$. It also follows that $\cosh z = 0$ if and only if $z = i(\pi/2 + n\pi)$ where $n \in \mathbb{Z}$, and $\sinh z = 0$ if and only if $z = in\pi$ where $n \in \mathbb{Z}$.

Definition. We define the other four hyperbolic trigonometric functions as:

$$\tanh z = \frac{\sinh z}{\cosh z},\ \coth z = \frac{\cosh z}{\sinh z},$$
$$\operatorname{sech} z = \frac{1}{\cosh z},\ \operatorname{csch} z = \frac{1}{\sinh z}.$$

Note 3.35.D. We have the following differentiation formulas:

$$\frac{d}{dz}[\tanh z] = \operatorname{sech}^2 z,\ \ \frac{d}{dz}[\coth z] = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}[\operatorname{sech} z] = -\operatorname{sech} z \tanh z,\ \ \frac{d}{dz}[\operatorname{csch} z] = -\operatorname{csch} z \coth z.$$

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