Section 3.36. Inverse Trigonometric and Hyperbolic Functions

Note. In this section we very briefly explore some inverse "functions." These are defined in terms of some of the other "multiple-valued" functions (namely, the square root and logarithm). When appropriately restricted to branches of square root and logarithm, the inverse functions are analytic on their domains.

Note. If we set $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$, then we have $(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$ and solving for e^{iw} using the quadratic equation (see Exercise 1.10.8(a); see Exercise 1.11.8(a) in the 9th edition) to solve for e^{iw} we get $e^{iw} = iz + (1 - z^2)^{1/2}$ (notice that $(1 - z)^{1/2}$ is double-valued). Applying the multiple-valued $\log z$ to both sides gives $x = -i \log(iz + (1 - z^2)^{1/2})$. This motivates the following definition.

Definition. For any $z \in \mathbb{C}$ define the inverse sine as the multiple-valued "function"

$$\sin^{-1} z = -i \log(iz + (1 - z^2)^{1/2})$$

where $(1 - z^2)^{1/2}$ is double-valued and "log" represents the multiple-valued logarithm function of Section 3.30.

Example. We have $\sin^{-1}(-i) = -i \log(1 \pm \sqrt{2})$. Since $1 + \sqrt{2} > 0$ then

$$\log(1 + \sqrt{2}) = \ln(1 + \sqrt{2}) + 2n\pi i \text{ where } n \in \mathbb{Z}$$

and since $1 - \sqrt{2} < 0$ then

$$\log(1 - \sqrt{2}) = \ln(\sqrt{2} - 1) + (2n + 1)\pi i \text{ where } n \in \mathbb{Z}.$$

Since $\sqrt{2} - 1 = (1 + \sqrt{2})^{-1}$ then $\ln(\sqrt{2} - 1) = -\ln(1 + \sqrt{2})$. So

$$\log(1 \pm \sqrt{2}) = (-1)^n \ln(1 + \sqrt{2}) + n\pi i \text{ where } n \in \mathbb{Z},$$

so

$$\sin^{-1} z = -i \log(1 \pm \sqrt{2}) = -i[(-1)^n \ln(1 + \sqrt{2}) + n\pi i] = -n\pi - i(-1)^n \ln(1 + \sqrt{2})$$

where $n \in \mathbb{Z}$.

Note. Similar computations to the one given above for the motivation of the definition of $\sin^{-1} z$, inspire the following definitions.

Definition. For any $z \in \mathbb{C}$ define the inverse cosine and inverse tangent as the multiple-valued "functions"

$$\cos^{-1} z = -i \log(z + i(1 - z^2)^{1/2}) \text{ and } \tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}.$$

Note. Notice that these are the same as the definitions in the real setting (though the domains are restricted in the real setting due to the presence of the square root and the logarithm).

Note. When specific branches of the square root function and the logarithm function are chosen, the resulting branches of the three inverse trig functions become analytic and we get the following derivatives:

$$\begin{aligned} \frac{d}{dz}[\sin^{-1} z] &= \frac{1}{(1-z^2)^{1/2}}, \\ \frac{d}{dz}[\cos^{-1} z] &= \frac{-1}{(1-z^2)^{1/2}}, \\ \frac{d}{dz}[\tan^{-1} z] &= \frac{1}{1+z^2}. \end{aligned}$$

Note. Similar arguments yield the following definitions of inverse hyperbolic trig functions:

$$\begin{aligned} \sinh^{-1} z &= \log(z + (z^2+1)^{1/2}), \\ \cosh^{-1} z &= \log(z + (z^2-1)^{1/2}), \\ \tanh^{-1} z &= \frac{1}{2}\log\frac{1+z}{1-z}. \end{aligned}$$

Revised: 1/15/2020