Chapter 4. Integrals

Section 4.37. Derivatives of Function $w(t)$

Note. When integrating a function of a complex variable, we will need to integrate on a path. This will involve a complex valued function of a real variable $w : \mathbb{R} \to \mathbb{C}$, denoted $w(t)$. Often the domain of w will be restricted to a closed and bounded interval in $\mathbb{R}$. When integrating over a path $w(t)$ we ultimately end up evaluating an integral of a single real variable (though the values of the integrand are complex). In this section we introduce $w(t)$ and its derivative.

Note 4.37.A. For w a complex valued function of a real variable t, in terms of the real and complex parts of w we have $w(t) = u(t) + iv(t)$, say. Since $w : \mathbb{R} \to \mathbb{C}$, we define

$$w'(t) = \lim_{h\to 0} \frac{w(t+h) - w(t)}{h},$$

so that

$$\begin{aligned} w'(t) &= \lim_{h\to 0} \frac{w(t+h) - w(t)}{h} = \lim_{h\to 0} \frac{(u(t+h) + iv(t+h)) - (u(t) + iv(t))}{h} \\ &= \lim_{h\to 0} \frac{u(t+h) - u(t)}{h} + i \lim_{h\to 0} \frac{v(t+h)) - v(t)}{h} = u'(t) + iv'(t), \end{aligned}$$

or $w'(t) = u'(t) + iv'(t)$ (provided all derivatives exist).

Note. We cannot use the derivative rules from Section 2.20 which involved complex valued functions of a *complex variable* and their derivatives. Two rules that apply in the current setting are the following:

$$\frac{d}{dt}[z_0 w(t)] = z_0 w'(t) \text{ and } \frac{d}{dt}[e^{z_0 t}] = z_0 e^{z_0 t}$$

where z_0 is a given constant. The second result can be established as:

$$\begin{aligned}
\frac{d}{dt}[e^{z_0 t}] &= \frac{d}{dt}[e^{x_0 t} e^{i y_0 t}] = \frac{d}{dt}[e^{x_0 t}\cos(y_0 t) + i e^{x_0 t}\sin(y_0 t)] \\
&= (e^{x_0 t}\cos(y_0 t))' + i(e^{x_0 t}\sin(y_0 t))' \\
&= [x_0 e^{x_0 t}]\cos(y_0 t) + e^{x_0 t}[-y_0 \sin(y_0 t)] + i\{[x_0 e^{x_0 t}]\sin(y_0 t) + e^{x_0 t}[y_0 \cos(y_0 t)]\} \\
&= x_0 e^{x_0 t}\cos(y_0 t) + i x_0 e^{x_0 t}\sin(y_0 t) + i\{y_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t))\} \\
&= x_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) + i y_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) \\
&= (x_0 + i y_0) e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) \\
&= (x_0 + i y_0) e^{x_0 t} e^{i y_0 t} = (x_0 + i y_0) e^{(x_0 + i y_0) t} \\
&= z_0 e^{z_0 t}.
\end{aligned}$$

Note. Other familiar differentiation rules hold in this setting, such as sum rules, product rules, quotient rules, and the Chain Rule. See Exercise 4.38.1 (Exercise 4.42.1(b) in the 9th edition). However, some of the common results from Calculus 1 do not carry over, as illustrated in the next example.

Example. Suppose that $w(t)$ is continuous on the interval $a \leq t \leq b$. Even if $w'(t)$ exists when $a < t < b$, the Mean Value Theorem for derivatives no longer applies. Consider $w(t) = e^{it}$ for $t \in [0, 2\pi]$. Then $w(0) = w(2\pi) = 1$, and the Mean Value Theorem would imply that there is some $t \in [0, 2\pi]$ for which $w'(t) = (w(2\pi) - w(0))/(2\pi - 0) = 0$ (technically, this is the special case called "Rolle's

Theorem"). However, $w'(t) = ie^{it}$ and so $|w'(t)| = |ie^{it}| = 1$, so the derivative never takes on the value 0. Hence the Mean Value Theorem does not hold in this setting.

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