Section 4.38. Definite Integrals of Functions $w(t)$

Note. In this section we define the integral of a function $w(t)$ of a real variable, use the Fundamental Theorem of Calculus to evaluate such integrals, and give a few examples.

Definition. For $w(t) = u(t)+iv(t)$, define the *integral* of $w(t)$ over interval $t \in [a,b]$ as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$

provided the integrals on the right exist.

Note 4.38.A. Recall that the Fundamental Theorem of Calculus says that the integral of a (real valued) continuous function can be found using antiderivatives. Since we define integrals of complex valued $w(t)$ in terms of integrals of real and imaginary parts (both of which are real valued functions of a real variable), then we can easily evaluate integrals provided we can find appropriate antiderivatives. That is, if $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous on the interval $t \in [a,b]$, if $W'(t) = w(t)$ for $t \in [a,b]$ (and so $U'(t) = u(t)$ and $V'(t) = v(t)$), then

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt = U(t)\Big|_a^b + iV(t)\Big|_a^b$$

$$= (U(b) + iV(b)) - (U(a) + iV(a)) = W(b) - W(a) = W(t)\Big|_a^b.$$

Example 4.38.2. As seen in Section 37, $\dfrac{d}{dt}[-ie^{it}] = e^{it}$, so

$$\int_0^{\pi/4} e^{it}\,dt = -ie^{it}\Big|_0^{\pi/4} = -ie^{i\pi/4} + i = -i(\cos(\pi/4) + i\sin(\pi/4) - 1)$$
$$= -i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

Note. We saw in the previous section that the Mean Value Theorem (for derivatives) may not hold in the current setting. It should come as no surprise that the Mean Value Theorem for Integrals may not hold either.

Example 38.3. Let $w(t) = e^{it}$. Then $w(t)$ is continuous for $t \in [0, 2\pi]$. Also

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt = -ie^{it}\Big|_0^{2\pi} = 0.$$

But the Mean Value Theorem of Integrals, if it held, would imply that there is $c \in [0, 2\pi]$ such that $w(c)(b-a) = \int_a^b w(t)\,dt$, or $w(c) = 0$. But $|w(t)| = |e^{it}| = 1$ for all $t \in [0, 2\pi]$, so there is no such c.

Revised: 1/19/2020