Section 4.39. Contours

Note. In this section we define a contour, which is the class of curves over which we will integrate complex valued functions of a complex variable.

Definition. A set of points $z = (x, y)$ in the complex plane is an *arc* if $x = x(t)$ and $y = y(t)$ for $t \in [a, b]$ where $x(t)$ and $y(t)$ are continuous functions of real parameter t. We denote an arc C as $z(t) = x(t) + iy(t)$ for $t \in [a, b]$.

Definition. Arc C is a *simple arc* (or a "Jordan arc") if it does not cross itself; that is, if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$ then C is a *simple closed curve* (or a "Jordan curve"). Such a curve is *positively oriented* when it is traced out in a counterclockwise direction as t ranges from a to b.

Example 4.39.2. The unit circle $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, is a simple closed positively oriented curve. So is the circle $z = z_0 + Re^{i\theta}$, $\theta \in [0, 2\pi]$.

Note. The parametric representation of a given arc (remember, the "arc" is the set of points in $\mathbb{C}$) is not unique. We could replace t with t^3 and adjust a and b accordingly, for example. In fact, we could define $t = \varphi(\tau)$ where $\tau \in [\alpha, \beta]$ and φ maps the interval $[\alpha, \beta]$ onto the interval $[a, b]$. See Figure 37.

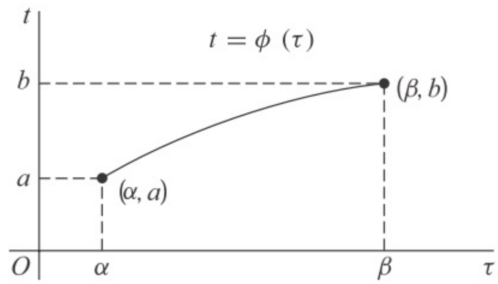

FIGURE 37

We assume φ is continuous with a continuous derivative and that $\varphi'(\tau) > 0$ for all $\tau \in [\alpha, \beta]$. With C as $z(t)$, $t \in [a, b]$, we can also express C as $Z(\tau) = z(\varphi(\tau))$, $\tau \in [\alpha, \beta]$.

Definition. Let arc C be expressed as $z(t) = x(t) + iy(t)$, $t \in [a, b]$. If $x'(t)$ and $y'(t)$ are continuous on $[a, b]$, the arc C is a *differentiable arc*. If arc C is a differentiable arc, the *length* of C is

$$L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Note. Since the integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ is continuous, then the integral defining the length of C exists. If we parameterize C in terms of τ where $t = \varphi(\tau)$ as above, then by Exercise 4.39.1(b) (Exercise 4.43.1(b) in the 9th edition of the book) we have the length of C as

$$L = \int_a^b |z'(t)|\, dt = \int_\alpha^\beta |z'(\varphi(\tau)|\varphi'(\tau)\, d\tau.$$

By Exercise 4.39.4 (Exercise 4.43.4 in the 9th edition of the book), the derivative of $Z(\tau)$ is $Z'(\tau) = z'(\varphi(\tau))\varphi'(\tau)$ so that the length of C is $L = \int_\alpha^\beta |Z'(\tau)|\, d\tau$. That is, the length of C is the same in both parameterizations of C.

Note. For differentiable arc C, if $z'(t) \neq 0$ for $t \in (a, b)$, then we have the unit tangent vector $\mathbf{T} = z'(t)/|z'(t)|$, where we interpret the complex number $z'(t)/|z'(t)|$ as a 2-dimensional vector tangent to arc C at point $z(t)$ and pointing in the direction of increasing t. You encounter this idea in Calculus 3 when considering tangent vectors, curvature, and normal vectors to a curve; see my online notes for Chapter 13. Vector-Valued Functions and Motion in Space.

Definition. A differentiable arc C given by $z(t)$, $t \in [a, b]$, is *smooth* if $z'(t)$ is continuous for $t \in [a, b]$ and $z'(t) \neq 0$ for $t \in (a, b)$. A *contour* (or "piecewise smooth arc") is an arc consisting of a finite number of smooth arcs joined end to end. When only the initial and final values of $z(t)$ are the same, contour C is a *simple closed contour*.

Note. There is a very plausible result from topology which states that a simple closed contour (or "simple closed curve") in a plane divides the plane into two disjoint regions, one bounded (the interior of the contour) and one unbounded (the exterior of the contour). This result is known as the Jordan Curve Theorem. For more information, see my online notes for Introduction to Topology (MATH 4357/5357); see Chapter 10. Separation Theorem in the Plane, and notice that this chapter includes a section on the Cauchy Integral Formula, which we will see in Section 50.

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