Section 4.40. Contour Integrals

Note. In this section we define the integral of a complex valued function of a complex variable along a contour.

Definition. Suppose that the equation $z = z(t)$, $t \in [a, b]$, represents contour C where $z_1 = z(a)$ and $z_2 = z(b)$. Suppose $f(z(t))$ is piecewise continuous for $t \in [a, b]$. The *contour integral* (or "line integral") of f along C is

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

Note 4.40.A. The contour integral of f is "linear":

$$\int_C (z_1 f(z) + z_2 g(z))\,dz = z_1 \int_C f(z)\,dz + z_2 \int_C g(z)\,dz.$$

Note 4.40.B. For contour C given by $z(t)$, $t \in [a, b]$, define $-C$ as $z(-t)$, $t \in [-b, -a]$. We then have, by Exercise 4.39.1(a) (Exercise 4.43.1(a) in the 9th edition of the book),

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\frac{d}{dz}[z(-t)]\,dt = -\int_{-b}^{-a} f(z(-t))z'(-t)\,dt$$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to t, evaluated at $-t$. If we substitute $\tau = -t$ we get

$$\int_{-C} f(z)\,dz = -\int_a^b f(z(\tau))z'(\tau)\,d\tau = -\int_C f(z)\,dz.$$

Note 4.40.C. Suppose a contour C consists of one contour C_1 followed by another contour C_2, see Figure 40. With C_1 given by $z(t)$, $t \in [a, c]$, and C_2 given by $z(t)$, $t \in [c, b]$, we have

$$\int_a^b f(z(t))z'(t)\,dt = \int_a^c f(z(t))z'(t)\,dt + \int_c^b f(z(t))z'(t)\,dt$$

or

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

For this reason, we may denote $C = C_1 + C_2$.

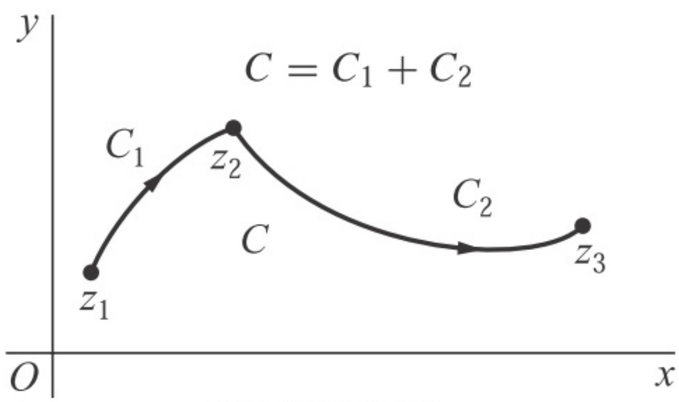

FIGURE 40

Note. On page 129, Brown and Churchill observe: "Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane." (See page 125 of the 9th edition of the book.)

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