Section 4.41. Some Examples

Note. In this section we consider several examples of integrals of a complex valued function of a complex variable along a contour.

Example 4.41.1. Let C be given by $z(\theta) = 2e^{i\theta}$, $\theta \in [-\pi/2, \pi/2]$ (the right half of the circle $|z| = 2$). See Figure 41. We now evaluate $I = \int_C \overline{z}\, dz$.

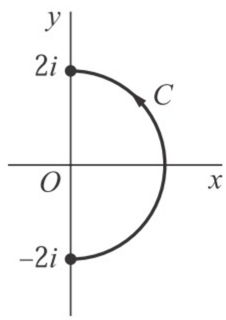

FIGURE 41

By definition of contour integral,

$$I = \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}} \frac{d}{d\theta}[2e^{i\theta}]\, d\theta = 4 \int_{-\pi/2}^{\pi/2} e^{-i\theta}(ie^{i\theta})\, d\theta = 4i \int_{-\pi/2}^{\pi/2} d\theta = 4\pi i.$$

Example 4.40.2. Let C_1 be the polygonal line OAB as in Figure 42. We evaluate the integral

$$I_1 = \int_{C_1} f(z)\, dz = \int_{OA} f(z)\, dz + \int_{AB} f(z)\, dz,$$

where $f(z) = f(x+iy) = y - x - i3x^2$. We parameterize OA as $z = 0 + iy$, $y \in [0,1]$ (so $z'(t) = z'(y) = i$) and AB as $z = x + i$, $x \in [0,1]$ (so $z'(t) = z'(x) = 1$). Notice that on OA (since $x = 0$) $f(z) = f(x+iy) = f(iy) = y$, and on AB (since $y = 1$)

$f(z) = f(x+iy) = 1 - x - i3x^2$. Then

$$\int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz = \int_{OA} (y)(i)\,dy + \int_{AB} (1 - x - i3x^2)(1)\,dx$$
$$= \frac{i}{2}y^2\Big|_0^1 + \left(x - \frac{1}{2}x^2\right)\Big|_0^1 - ix^3\Big|_0^1 = \frac{i}{2} + \frac{1}{2} - i = \frac{1-i}{2}.$$

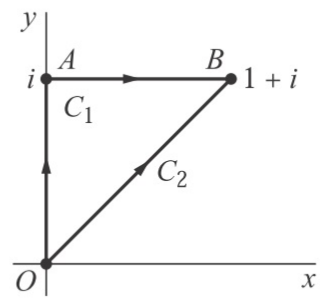

FIGURE 42

Let C_2 be the line segment OB as in Figure 42. We parameterize OB as $z = x + ix$, $x \in [0,1]$ (so $z'(t) = z'(x) = 1 + i$). On OB (since $y = x$) $f(z) = f(x+iy) = x - x - i3x^2 = -i3x^2$. So another integral is

$$I_2 = \int_{C_2} f(z)\,dz = \int_0^1 (-i3x^2)(1+i)\,dx = 3(1-i)\int_0^1 x^2\,dx = (1-i).$$

Notice that $I_1 \neq I_2$, even though the integrals are along contours that both start at O and end at B. So, in this example at least, there is some type of path dependence on the value of a contour integral (recall, from the Fundamental Theorem of Calculus, that a real integral only depends on the starting point a, the ending point b, and an antiderivative). In fact, we can integrate around the simple closed contour $C_1 - C_2$ (along the closed polygonal line $OABO$) to get

$$\int_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = I_1 - I_2 = \frac{-1+i}{2}.$$

We will often get integrals to have values of 0 over simple closed contours. This will be addressed in detail in Section 48.

Example 4.41.3. In the previous example, we saw that the value of a contour integral *can* depend the path and not just on the endpoints of the path. We now consider an example where we do have path independence. Consider $f(z) = z$ and let C be any smooth arc given by, say, $z(t)$, $t \in [a, b]$, where $z(a) = z_1$ and $z(b) = z_2$. We then have

$$\int_C z\,dz = \int_a^b z(t)z'(t)\,dt = \int_a^b \frac{d}{dt}\left[\frac{(z(t))^2}{2}\right]\,dt$$

$$= \left.\frac{(z(t))^2}{2}\right|_a^b = \frac{(z(b))^2 - (z(a))^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

We can generalize this example to a contour C which consists of piecewise smooth arcs $C_1, C_2, \ldots, C_n$ joined end to end. With z_k and z_{k+1} as the endpoints of C_k, we have

$$\int_C z\,dz = \int_{C_1+C_2+\cdots+C_n} z\,dz = \sum_{k=1}^n \int_{C_k} z\,dz$$

$$= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} z\,dz = \sum_{k=1}^n \frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}.$$

Therefore the value of the integral depends only on the endpoints (z_1 and z_2 in the first case, z_1 and z_{n+1} in the second case). The reason this is the case is related to the fact that the function $f(z) = z$ has an antiderivative valid in the entire complex plane.

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