Section 4.42. Examples with Branch Cuts

Note. In this section we consider examples of integrals of a complex valued function involving branches of the logarithm.

Example 4.42.1. Let C be the semicircular path $z = 3e^{i\theta}$, $0 \le \theta \le \pi$ (see Figure 44), and consider the branch of the square root function $f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$ where $|z| > 0$ and $0 < \arg z < 2\pi$. Notice that f is not defined at $z = 3 \in C$ but we will see that the contour integral is defined on C (similar to how we might integrate in the real setting up to a vertical asymptote of a real function; this is an example of an improper integral).

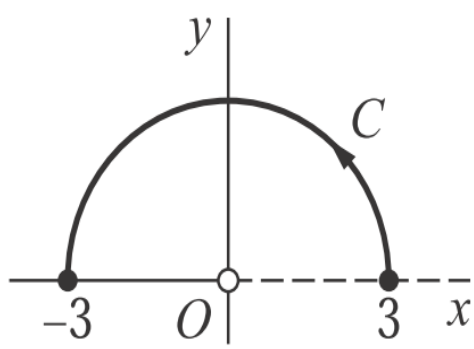

FIGURE 44

Now

$$f(z(\theta)) = \exp\left(\frac{1}{2}\log(3e^{i\theta})\right) = \exp\left(\frac{1}{2}(\ln 3 + i\theta)\right)$$
$$= \exp\left(\frac{1}{2}\ln 3\right)\exp\left(\frac{1}{2}i\theta\right) = \sqrt{3}e^{i\theta/2}$$

where $\theta \in (0, \pi]$. Since $z'(\theta) = 3ie^{i\theta}$ then

$$f(z(\theta))z'(\theta) = (\sqrt{3}e^{i\theta/2})(3ie^{i\theta}) = 3\sqrt{3}ie^{i3\theta/2} = i3\sqrt{3}\cos\frac{3\theta}{2} - 3\sqrt{3}\sin\frac{3\theta}{2}$$

where $\theta \in (0, \pi]$. We then have

$$\begin{aligned}
\int_C z^{1/2}\,dz &= \int_0^\pi f(z(\theta))z'(\theta)\,d\theta = \int_0^\pi \left(i3\sqrt{3}\cos\frac{3\theta}{2} - 3\sqrt{3}\sin\frac{3\theta}{2}\right)\,d\theta \\
&= i\int_0^\pi \left(3\sqrt{3}\cos\frac{3\theta}{2}\right)\,d\theta - \int_0^\pi \left(3\sqrt{3}\sin\frac{3\theta}{2}\right)\,d\theta \\
&= i\lim_{a\to 0^+}\int_a^\pi \left(3\sqrt{3}\cos\frac{3\theta}{2}\right)\,d\theta - \lim_{a\to 0^+}\int_a^\pi \left(3\sqrt{3}\sin\frac{3\theta}{2}\right)\,d\theta \\
&= \lim_{a\to 0^+}\left(i2\sqrt{3}\sin\frac{3\theta}{2} + 2\sqrt{3}\cos\frac{3\theta}{2}\right)\Big|_a^\pi \\
&= \left(i2\sqrt{3}\sin\frac{3\pi}{2} + 2\sqrt{3}\cos\frac{3\pi}{2}\right) - \left(i2\sqrt{3}\sin 0 + 2\sqrt{3}\cos 0\right) \\
&= -i2\sqrt{3} - 2\sqrt{3} = -2\sqrt{3}(i+1).
\end{aligned}$$

Notice that the nonexistence of the integrand at $z = 3$ disappeared in the computation with the limit and the fact that the real and imaginary parts have antiderivatives valid at $z = 3$ (or $\theta = 0$ as it appears in the computation). But the branch of $f(z) = z^{1/2}$ used here also has an antiderivative defined on C except at $z = 3$. That is, f is piecewise continuous on C if we extend the definition of f to $z = 3$ by defining $f(3) = 3\sqrt{3}i$ (the value of an integral is not affected by changing the value of the integrand at a single point, and this change gives us the piecewise continuity we need to use complex antiderivatives) and so we can more quickly compute:

$$\int_C f(z)\,dz = \int_0^\pi f(z(\theta))z'(\theta)\,d\theta = 3\sqrt{3}i\int_0^\pi e^{i3\theta/2}\,d\theta$$

$$= 2\sqrt{3}e^{i3\theta/2}\Big|_0^\pi = 2\sqrt{3}(e^{3i\pi/2} - 1) = -2\sqrt{3}(i+1).$$

However, we will largely continue to evaluate such integrals using limits and giving rigorous computations.

Example 4.42.2. Suppose that C is the positively oriented circle $z = Re^{i\theta}$, $-\pi \le \theta \le \pi$ (see Figure 45). Let a denote any nonzero real number. Using the principal branch of z^{a-1} we have $f(z) = z^{a-1} = \exp((a-1)\operatorname{Log} z)$, $|z| > 0$ and $-\pi < \operatorname{Arg} z < \pi$.

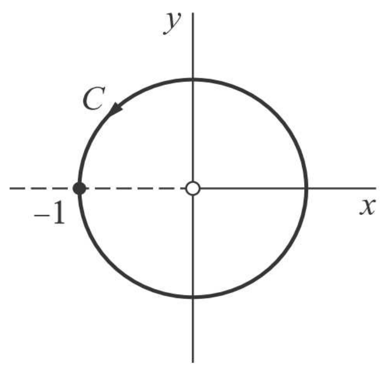

FIGURE 45

We now consider $\int_C z^{a-1}\,dz$. We have

$$f(z(\theta))z'(\theta) = (Re^{i\theta})^{a-1}(Rie^{i\theta}) = iR^a e^{ia\theta} = iR^a \cos a\theta - R^a \sin a\theta.$$

Notice that $f(z(\theta))z'(\theta)$ is continuous for $\theta \in (-\pi, \pi)$, so that it is piecewise continuous on $[-\pi, \pi]$ (by extending f at the endpoints to get continuity, as in the previous example). So

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} f(z(\theta))z'(\theta)\,d\theta = iR^a \int_{-\pi}^{\pi} e^{ia\theta}\,d\theta$$

$$= iR^a \frac{e^{ia\theta}}{ia}\bigg|_{-\pi}^{\pi} = \frac{R^a}{a}(e^{ia\pi} - e^{-ia\pi}) = \frac{R^a}{a}(2i\sin(a\pi)).$$

This holds for all nonzero real a. When a is a nonzero integer (say $a = n+1 \in \mathbb{Z}$, we get $\int_C z^n\,dz = \frac{R^{n+1}}{n+1} 2i\sin((n+1)\pi) = 0$. Brown and Churchill observe that “if a is allowed to be zero” (which it is not!), then $\int_C \frac{1}{z}\,dz = 2\pi i$. This result will hold later, but not from the computation performed here.