Section 4.43. Upper Bounds for Moduli of Contour Integrals

Note. Recall from Calculus 1 that for Riemann integrable function f on $[a,b]$ where $|f(x)| \le M$ for all $x \in [a,b]$, we have $\int_a^b f(x)\,dx \le M(b-a)$. In this section we prove a similar result for complex contour integrals and give applications.

Lemma 4.43.A. If $w(t)$ is a piecewise continuous complex valued function defined on an interval $a \le t \le b$, then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt.$$

Theorem 4.43.A. Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defined, then $\left|\int_C f(z)\,dz\right| \le ML$.

Example 4.43.1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$. consider $\displaystyle\int_C \frac{z+4}{z^3+1}\,dz$. For $z \in C$, we have $|z| = 2$ so $|z+4| \le |z| + 4 = 6$ and $|z^3+1| \ge ||z|^3 - 1| = 7$ (by Corollary 1.4.1). So for $z \in C$ we have

$$|f(z)| = \left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \le \frac{6}{7} = M.$$

Now L is the length of 1/4 of a circle of radius 2, so $L = 2\pi(2)/4 = \pi$. Therefore, by Theorem 4.43.A,

$$\left|\int_C \frac{z+4}{z^3+1}\,dz\right| \le ML = \frac{6\pi}{7}. \qquad \square$$

Example 4.43.2. Let $C_R = \{z = Re^{i\theta} \mid \theta \in [0,\pi]\}$ and let $z^{1/2}$ denote the branch of the square root function

$$z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \text{ where } \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

See Figure 47.

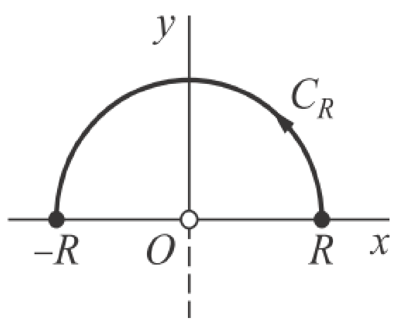

FIGURE 47

Find a bound on $\displaystyle\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz$ where $R > 1$ and show $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz = 0$.

Solution. With $z \in C_R$ we have $|z| = R > 1$ and so $|z^{1/2}| = |\sqrt{R}e^{i\theta/2}| = \sqrt{R}$ and $|z^2+1| \geq ||z|^2 - 1| = R^2 - 1$ (by Corollary 1.4.1). So for $z \in C_R$ we have

$$|f(z)| = \left|\frac{z^{1/2}}{z^2+1}\right| \leq \frac{\sqrt{R}}{R^2-1} = M_R.$$

Now L is the length of 1/2 of a circle of radius R, so $L = 2\pi(R)/2 = \pi R$. So by Theorem 4.43.A,

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \leq M_R L = \frac{\sqrt{R}}{R^2-1}\pi R = \frac{\pi R^{3/2}}{R^2-1}.$$

So

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \leq \lim_{R\to\infty} \frac{\pi R^{3/2}}{R^2-1} = 0. \quad \square$$