Section 4.44. Antiderivatives

Note. Recall that the Fundamental Theorem of Calculus, Part 2 states:

> If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then
> $$\int_a^b f(x)\,dx = F(b) - F(a).$$

In this section we see that the same result does not exactly hold in the complex setting. The complex result is heavily related to the existence of an antiderivative, though.

Theorem 4.44.A. Suppose that a function $f(z)$ is continuous on a domain D. The following are equivalent:

(a) $f(z)$ has an antiderivative $F(z)$ throughout D;

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely
$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$
where $F(z)$ is the antiderivative in statement (a);

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

Note. We prove Theorem 4.44.A in the next section and illustrate its application in this section.

Example 4.43.A. If p is a polynomial function, then it has an antiderivative P such that $P' = p$ throughout the entire complex plane. So, by Theorem 4.44.A part (b), for any contour joining $z_1, z_2 \in \mathbb{C}$ we have $\int_{z_1}^{z_2} p(z)\,dz = P(z_2) - P(z_1)$.

Example 4.44.2. Consider $f(z) = 1/z^2$ and $C = \{z = 2e^{i\theta} \mid \theta \in [-\pi, \pi]\}$. We have $F(z) = -1/z$ is an antiderivative of f valid on $\mathbb{C}\setminus\{0\}$. So, by Theorem 4.44.A part (b), $\displaystyle\int_C \frac{dz}{z^2} = 0$ since C starts and stops at the same point. In fact, we would expect many integrals along closed contours to be 0. This is not the case for all functions, though.

Example 4.44.3. Consider $f(z) = 1/z$. Notice that f does not have an antiderivative which is valid on any domain containing the contour $C = \{z = 2e^{i\theta} \mid \theta \in [-\pi, \pi]\}$, due to the presence of the branch cut. If Theorem 4.44.A did apply, then the integral would have to be 0 by part (c) of the theorem. We now show that the integral is not 0. Let $C_1 = \{z = 2e^{i\theta} \mid \theta \in [-\pi/2, \pi/2]\}$; see Figure 50. The principal branch of the logarithm, $\mathrm{Log}\, z = \ln r + i\Theta$ where $r > 0$ and $\Theta \in (-\pi, \pi)$, is an antiderivative of f which is valid on domain $D = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ containing C_1. So

$$\int_{C_1} \frac{dz}{z} = \mathrm{Log}\, z \Big|_{-2i}^{2i} = \mathrm{Log}\,(2i) - \mathrm{Log}\,(-2i) = \left(\ln 2 + i\frac{\pi}{2}\right) - \left(\ln 2 - i\frac{\pi}{2}\right) = \pi i.$$

Let $C_2 = \{z = 2e^{i\theta} \mid \theta \in [\pi/2, 3\pi/2]\}$; see Figure 51. The branch of the logarithm, $\log z = \ln r + i\theta$ where $r > 0$ and $\theta \in (0, 2\pi)$, is an antiderivative of f which is valid on domain $D = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 0\}$ containing C_2. So

$$\int_{C_2} \frac{dz}{z} = \log z \Big|_{2i}^{-2i} = \log(-2i) - \log(2i) = \left(\ln 2 + i\frac{3\pi}{2}\right) - \left(\ln 2 + i\frac{\pi}{2}\right) = \pi i.$$

We now see that

$$\int_C \frac{dz}{z} = \int_{C_1+C_2} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

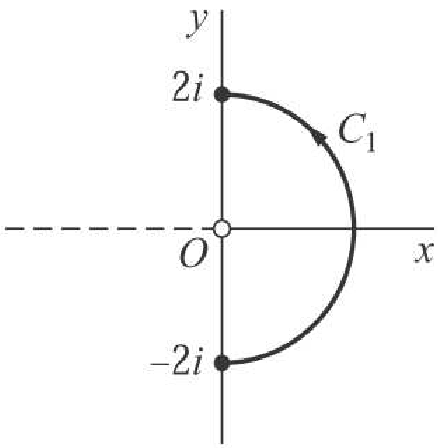

FIGURE 50

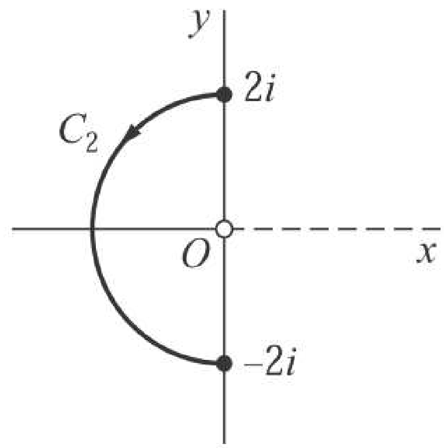

FIGURE 51

Example 4.44.4. Consider $f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2}$, where $r > 0$ and $\theta \in (0, 2\pi)$, and let C_1 be any contour from $z = -3$ to $z = 3$ that lies in the open upper half plane, except for its endpoints; see Figure 52. Notice that $f(z)$ is not defined at the endpoint $z = 3$, but this does not affect the integral as argued in Section 42. In this example, Brown and Churchill replace the given branch $f(z)$ of $z^{1/2}$ with a branch $f_1(z)$ which equals the given branch at all points on the contour, except that the new branch is defined at $z = 3$: $f_1(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2}$, where $r > 0$ and $\theta \in (-\pi/2, 3\pi/2)$. Since $f_1(z)$ equals $f(z)$, except at a single point, on the contour, then the integrals will be the same since a limit as $z \to 3$ along C_1 is the same for both $f(z)$ and $f_1(z)$ (and similarly for antiderivatives $F(z) = F_1(z)$, respectively). But $f_1(z)$ has an antiderivative valid on a domain containing C_1, so that Theorem 4.44.A can be used. An antiderivative for $f_1(z)$ is $F_1(z) = \frac{2}{3}z^{3/2} = \frac{2}{3}r^{3/2}e^{i3\theta/2}$, where $r > 0$ and $\theta \in (-\pi/2, 3\pi/2)$. Therefore, by Theorem 4.44.A,

$$\int_C f(z)\,dz = \int_{C_1} f_1(z)\,dz = F_1(z)\Big|_{-3}^{3} = F_1(z)\Big|_{3e^{i\pi}}^{3e^{i0}}$$

$$= \frac{2}{3}3^{3/2}e^{i3(0)/2} - \frac{2}{3}3^{3/2}e^{i3(\pi)/2} = 2\sqrt{3}(e^{i0} - e^{i3\pi/2}) = 2\sqrt{3}(1+i).$$

The fact that we can evaluate the integral by knowing very little about the contour from $z = -3$ to $z = 3$, fore shadows the Cauchy Integral Formula in Section 50.

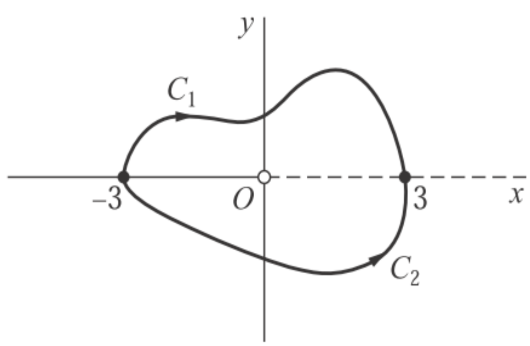

FIGURE 52

Revised: 4/14/2020