Section 4.45. Proof of the Theorem

Note. We now prove the result from the previous section. We start by restating the result.

Theorem 4.44.A. Suppose that a function $f(z)$ is continuous on a domain D. The following are equivalent:

(a) $f(z)$ has an antiderivative $F(z)$ throughout D;

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

Proof. First we show (a) $\Longrightarrow$ (b). Suppose $f(z)$ has an antiderivative $F(z)$ on the domain D. Let C be a contour from z_1 and z_2 that is *smooth*, lies in D and has parametric representation $z = z(t)$ where $a \le t \le b$. Then by Exercise 4.39.5 we have

$$\frac{d}{dt}[F(z(t))] = F'(z(t))z'(t) = f(z(t))z'(t) \text{ where } a \le t \le b.$$

So

$$\begin{aligned}\int_C f(z)\,dz &= \int_Z f(z(t))z'(t)\,dt \text{ by definition (see Section 4.40)}\\ &= F(z(t))|_{t=a}^{t=b} = F(z(b)) - F(z(a)) \text{ by Note 4.38.A}\\ &= F(z_2) - F(z_1) \text{ since } z_1 = z(a) \text{ and } z_2 = z(b)\end{aligned}$$

So (b) holds in the event that C is smooth. Now a contour is piecewise smooth by definition (see Section 4.39), so for C any contour that is piecewise smooth, say C consists of the n smooth contours $C_1, C_2, \ldots, C_n$ (with C_1 a smooth contour from $z(a) = z_1$ to z_2, C_2 a smooth contour from z_2 to z_3, $\ldots$, and C_n a smooth contour from z_n to $z_{n+1} = z(b)$), then

$$\begin{aligned}\int_C f(z)\,dz &= \sum_{k=1}^{n} \int_{C_k} f(z)\,dz \text{ by Note 4.40.C and induction}\\ &= \sum_{k=1}^{n} (F(z_{k+1}) - F(z_k)) \text{ by the proof above, since each } C_k \text{ is smooth}\\ &= F(b) - F(a).\end{aligned}$$

That is, (b) holds.

Next, we show (b) $\Longrightarrow$ (c). Suppose that integration of $f(z)$ is independent of the contour in D and instead only depends on the endpoints of the contour. Let C be any closed contour in D and let z_1 and z_2 be two distinct points on C. Form paths C_1 and C_2 (see Figure 53).

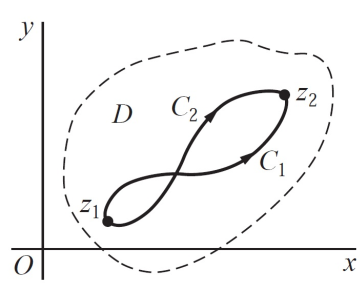

Figure 53

Since we hypothesize that the values of integrals are independent of contours, then we have $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ or, by Notes 4.40.B and 4.40.C,

$$0 = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz$$
$$= \int_{C_1 - C_2} f(z)\,dz = \int_C f(z)\,dz.$$

So integrals of $f(z)$ around closed contours lying entirely in D all have value zero and (c) holds.

Finally, we show (c) $\Longrightarrow$ (a). Suppose integrals of $f(z)$ around closed contours lying entirely in D all have value zero. Let C_1 and C_2 denote any two contours lying in D from a point z_1 to a point z_2. Then $C = C_1 - C_2$ is a closed contour in D and so by hypothesis,

$$\begin{aligned} 0 &= \int_C f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz \text{ by Note 4.40.C} \end{aligned}$$

and so $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ (in fact, we have shown that (c) $\Longrightarrow$ (b) here). Let $z_0 \in D$ and define function $F(z)$ as $F(z) = \int_{z_0}^{z} f(s)\,dz$ where $z \in D$. The path independence of integrals shows that F is *well-defined*. We now show that $F'(z) = f(z)$ on D. Let $z + \Delta z$ be any point distinct from z and lying in some neighborhood of z that is small enough to be contained in D (such a neighborhood exists since D is hypothesized to be open). Then

$$\begin{aligned} F(z + \Delta z) - F(z) &= \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds \\ &= \int_{z}^{z+\Delta z} f(s)\,ds \text{ by Note 4.40.C.} \end{aligned}$$

Since Δz lies in a neighborhood of z then we see that Δz may be selected as a line segment (see Figure 54).

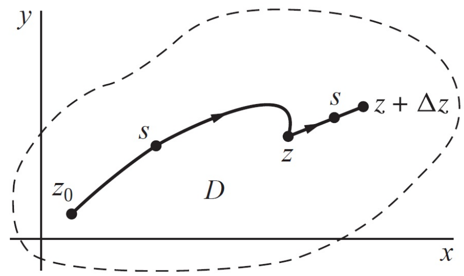

Figure 54

Since $\int_z^{z+\Delta z} ds = \Delta z$ by Exercise 4.42.5 (Exercise 4.46.5 in the 9th edition of the book), we have

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\,ds = \frac{f(z)}{\Delta z}\int_z^{z+\Delta z} ds = f(z).$$

So

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{\int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds}{\Delta z} - f(z)$$

$$= \frac{1}{\Delta z}\left(\int_z^{z+\Delta z} f(s)\,ds\right) - \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\,ds = \frac{1}{\Delta z}\int_z^{z+\Delta z} (f(s)-f(z))\,ds.$$

Since f is continuous at z by hypothesis, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(s) - f(z)| < \varepsilon$ whenever $|s - z| < \delta$. Consequently, if $|\Delta z| < \delta$ then

$$\begin{aligned}\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| &= \left|\frac{1}{\Delta z}\int_z^{z+\Delta z} (f(s)-f(z))\,ds\right| \\ &< \frac{1}{|\Delta z|}\varepsilon|\Delta z| \text{ by Theorem 4.43.A} \\ &= \varepsilon.\end{aligned}$$

So by the definition of limit (see Section 2.15),

$$\lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z).$$

That is, $F'(z) = f(z)$ and so (a) holds. ∎