Section 4.46. Cauchy-Goursat Theorem

Note. We saw in Section 44.4, Theorem 4.44.A(c), that if f is continuous throughout a domain D and has an antiderivative on D, then $\int_C f(z)\,dz = 0$ for any closed contour C in D. In this section, we state a result giving other conditions on a function f which guarantees that the integral of f around a simple closed contour is zero.

Note 4.46.A. For C the contour $z = z(t)$, $t \in [a, b]$, we have by definition (see Section 4.40)

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt,$$

and if $f(z) = u(x, y) + iv(x, y)$ where $z(t) = x(t) + iy(t)$ then

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b (u(x(t), y(t)) + i(v(x(t), y(t))(x'(t) + iy'(t))\,dt \\
&= \int_a^b (u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t))\,dt \\
&\quad + i\int_a^b (u(x(t), y(t))y'(t) + v(x(t), y(t))\,x'(t))\,dt \\
&= \int_a^b (ux' - vy')\,dt + i\int_a^b (uy' + vx')\,dt.
\end{aligned}$$

With $dx = x'\,dt$ and $dy = y'\,dt$ we have in terms of functions of x and y,

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx). \qquad (3)$$

Note. Recall from vector calculus:

> **Green's Theorem.** Let $P(x,y)$ and $Q(x,y)$ be continuous and suppose each of the first partial derivatives of P and Q are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. Then
>
> $$\int_C (P(x,y)\,dx + Q(x,y)\,dy) = \iint_R (Q_x - P_y)\,dA.$$

Note 4.46.B. With f analytic in R (interpreting R as a region in $\mathbb{C}$ instead of $\mathbb{R}^2$ in the obvious way), then f is differentiable on R. **If** the derivative f' is continuous on R then the first-order partial derivatives of u and v are continuous since $f' = u_x + iv_x = v_y - iu_y$ by Theorem 2.22.A (The Cauchy-Riemann Equations and Continuity Imply Differentiability). So by Green's Theorem, equation (3) becomes

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\\ &= \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA \qquad (4)\end{aligned}$$

with $u(x,y) = P(x,y)$ and $v(x,y) = -Q(x,y)$ in the integral $\int_C (u\,dx - v\,dy)$ and with $v(x,y) = P(x,y)$ and $u(x,y) = Q(x,y)$ in the integral $\int_C (v\,dx + u\,dy)$. By the Cauchy-Riemann equations (Theorem 2.22.A), $u_x = v_y$ and $u_y = -v_x$, so

$$\int_C f(z)\,dz = \iint_R (-v_x - (-v_x))\,dA + i\iint_R ((v_y) - v_y)\,dA = \iint_R 0\,da = 0.$$

So when f is analytic on R and f' is continuous on R then $\int_C f(z)\,dz = 0$.

Note. Édouard Goursat (1858–1936) was the first to prove that the condition of continuity of f' can be removed. (We will see later that an analytic function on a region has a power series representation and hence is infinitely differentiable and "infinitely continuous"; see Section 5.57, "Taylor Series").

Note. Goursat's observation allows to prove the following.

Theorem 4.46.A. *The Cauchy-Goursat Theorem.*
If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)\, dz = 0$.

Note. Édouard Goursat (May 21, 1858–November 25, 1936) was an influential French analyst of the late 19th and early 20th centuries. He studied at the École Normale Super'eieure in Paris starting in 1876. He was influenced by Emile Picard, Jean Darboux, and Charles Hermite. Cauchy had proved that the integral of a function of a complex variable over a simple closed contour is zero if the function has a continuous derivative inside the contour. Goursat removed the condition of a continuous derivative inside the contour and simply required that the function be differentiable inside the contour. He published his results in *Démonstration dy théorèm de Cauchy* in 1884, leading to the name of "The Cauchy-Goursat Theorem" for Theorem 4.46.A. As some trivia, Goursat named L'Hôpital's rule in his *Cours d'analyses Mathématique.* The result appears in earlier texts, but we call the result "L'Hôpital's Rule" today due to Goursat's text. These historical notes and the photo are from the "MacTutor History of Mathematics archive" biography of Goursat.

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