Section 4.47. Proof of the Theorem

Note. In this section we give a proof of the Cauchy-Goursat Theorem (Theorem 4.46.A):

> If a function f is analytic at all points interior to and on a simple closed contour C, then
> $$\int_C f(z)\,dz = 0.$$

Note. The proof is involved and the version given in Brown and Churchill lacks rigor in places. The first step is to show that f' is continuous, which is done in Lemma 4.47.1. Second, we put a bound on $\int_C f(z)\,dz$, which is done in Lemma 4.47.A, and third we show that the bound can be made arbitrarily small. This is structurally the same as the proof given in Complex Analysis 1/2 (MATH 5510/5520); see my online notes on IV.8. Goursats Theorem. However, here we show that f' is continuous based on an argument involving "little squares," whereas "little triangles" are used in the proof given in graduate Complex Analysis.

Note. We start by covering the region R, which consists of the points on and inside C, with little squares which have edges parallel to either the real axis or the imaginary axis. We refer to squares which are subsets of R as "squares" and the intersection of a square with R which does not include all the points in the square as a "partial square." Since R is bounded, this results in a covering of R by a finite number of squares and partial squares. See Figure 55. Notice that we don't require all of the squares to be the same size.

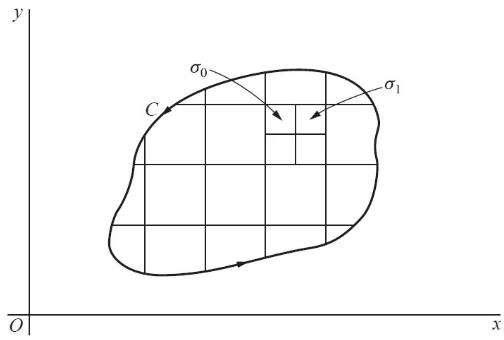

FIGURE 55

Lemma 4.47.1. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any $\varepsilon > 0$, the region R can be covered with a finite number of squares and partial squares indexed by $j = 1, 2, \ldots, n$ such that in each one there is a fixed point z_j for which the inequality

$$\left| \frac{f(z) - f(z_j)}{z - z_i} - f'(z_j) \right| < \varepsilon$$

is satisfied by all points other than z_j in that square or partial square.

Lemma 4.47.A. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any covering of R by squares and partial squares as described in Lemma 4.47.1, put positive orientations on each of the boundaries of the squares and partial squares (see Figure 57) and denote the resulting positively oriented contours as $C_1, C_2, \ldots, C_n$. On the jth square or partial square, define

$$\delta_j(z) = \begin{cases} (f(z) - f(z_j))/(z - z_j) - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j. \end{cases}$$

Then

$$\left| \int_C f(z)\,dz \right| \le \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)\,dz \right|.$$

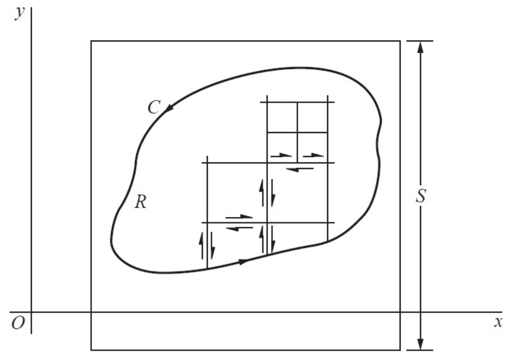

FIGURE 57

Note. We now have the equipment to give a proof of the Cauchy-Goursat Theorem.

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