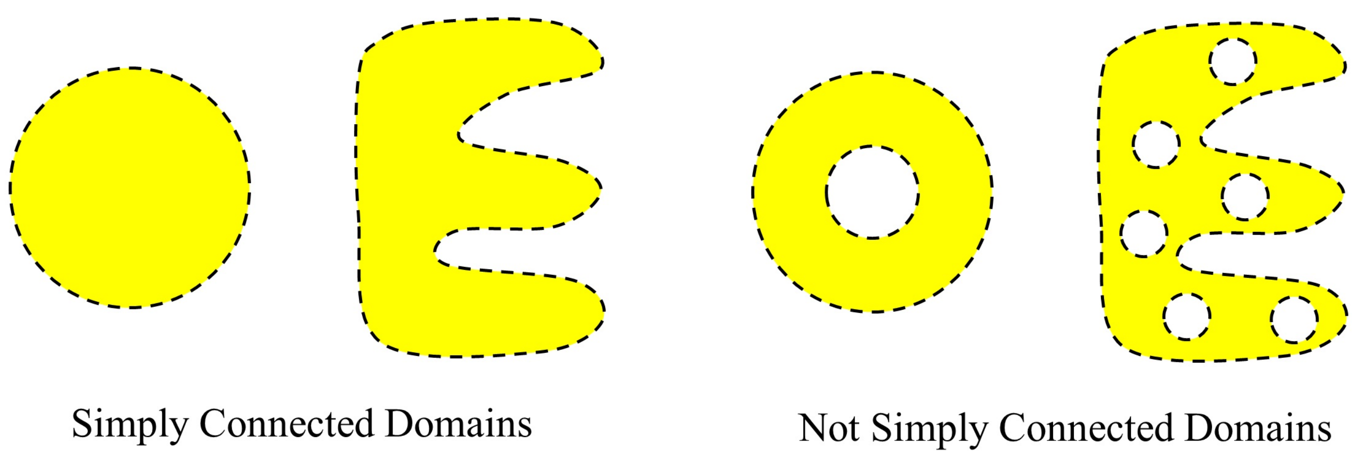
Section 4.48. Simply Connected Domains

Note. Informally, a simply connected domain is an open connected set with "no holes." The main result in this section, similar to the Cauchy-Goursat Theorem (Theorem 4.44.A), states that an integral of a function analytic over a simply connected domain is 0 for all closed contours in the domain.

Definition. A *simply connected domain* D is a domain such that every simple closed contour in the domain encloses only points in D.

Note. We have:



Theorem 4.48.A. If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)\,dz = 0$ for every closed contour C lying in D.

Note. The proof of Theorem 4.48.A for some more general contours (those which might intersect themselves an infinite number of times) is rather involved. In graduate Complex Analysis 1 and 2 (MATH 5510 and 5520), Theorem 4.48.A is titled "Cauchy's Theorem (Fourth Version)" (see Theorem IV.6.15 in IV.6. The Homotopic Version of Cauchys Theorem and Simple Connectivity), though the proof is delayed until Section VIII.2. Simple Connectedness in Theorem VIII.2.2 (see the proof that (a) implies (e)).

Example. If C denotes any closed contour lying in the open disk $|z| < 2$ then $\displaystyle\int_C \frac{ze^z}{(z^2+9)^5}\,dz = 0$. This is because the open disk $|z| < 2$ is a simply connected domain and the integrand is analytic on the disk (the integrand is analytic except at $\pm 3i$), so that Theorem 4.48.A yields the value of 0 for the integral.

Corollary 4.48.B. A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Note. Since the complex plane $\mathbb{C}$ is simply connected, Corollary 4.48.A implies that every entire function has an antiderivative throughout $\mathbb{C}$.

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