Section 4.49. Multiply Connected Domains

Note. In this section, we extend the Cauchy-Goursat Theorem to more general domains than simply connected ones (under certain hypotheses).

Definition. A domain D that is not simply connected is a *multiply connected domain*. That is, domain D is multiply connected if there is a simple closed contour in D which encloses points in $\mathbb{C} \setminus D$.

Theorem 4.49.A. Suppose that

(a) C is a simple closed contour, parameterized in the counterclockwise direction; and

(b) C_k, for $k = 1, 2, \ldots, n$, are simple closed contours interior to C, all parameterized in the clockwise direction, that are disjoint and whose interiors have no points in common (so one such contour cannot be inside another).

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k, then

$$\int_c f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

Note. As seen in the "proof" of Theorem 4.49.A, we are skipping many details on the behavior of simple closed curves and open connected sets (i.e., domains). Even in our use of the term "interior" of a simple closed curve, we are appealing to an intuitive idea. It is in the Jordan Curve Theorem that it is originally, rigorously established that a simple closed curve separates the plane into two components—one bounded (called the interior of the curve) and one unbounded (the exterior of the curve). This result belongs to the area of math called topology (more specifically, algebraic topology). For details on these ideas, see Chapter 10. Separation Theorems in the Plane, in my online notes for Introduction to Topology (MATH 4357/5357) and James Munkres' *Topology*, 2nd edition (Prentice Hall, 2000).

Note. The existence of the polygonal paths L_k, $1 \leq k \leq n+1$, are guaranteed by a result presented in the graduate-level Complex Analysis 1 (MATH 5510) class: "An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a, b \in G$, there is a polygon from a to b lying entirely inside of G." This is Theorem II.2.3 in Conway's *Functions of One Complex Variable I*, 2nd edition (Springer, 1978). See also my online notes for Complex Analysis 1 (MATH 5510) on II.2. Connectedness.

Corollary 4.49.B. Principle of Deformation.

Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Note. Corollary 4.49.B is called the "Principle of Deformation" for the following reason. The idea is that C_1 can be continuously deformed into C_2, always passing through points at which f is analytic. This relationship between C_1 and C_2 is usually called "homotopy," so that Corollary 4.49.B can be restated as: "If simple closed contours C_1 and C_2 are homotopic over a region on which function f is analytic, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$." This is stated in Conway's book as Theorem IV.6.7, Cauchy's Theorem (Third Version); Conway does not require the contours to be simple. See also my online notes for Complex Analysis 1 (MATH 5510) on IV.6. The Homotopic Version of Cauchys Theorem and Simple Connectivity.

Example. Let C be any positively oriented simple closed contour with 0 as an interior point. Then to evaluate $\displaystyle\int_C \frac{dz}{z}$, we can consider the simple closed contour C_0 as the unit circle in $\mathbb{C}$ positively oriented. Since $f(z) = 1/z$ is analytic on $\mathbb{C}\setminus\{0\}$ and C can be continuously deformed to C_0, then by Corollary 4.49.B,

$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z} = 2\pi i$$

(by Example 42.2).

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