Section 4.50. Cauchy Integral Formula

Note. In this section we prove a result which we can use to evaluate certain integrals. We extend the result in the next section and the extension will be of great theoretical interest.

Theorem 4.50.A. Cauchy Integral Formula.
Let f be analytic everywhere inside and on simple closed contour C, parameterized in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}.$$

Example. Let C be the positively oriented circle $|z| = 2$ and consider $\displaystyle\int_C \frac{z\,dz}{(9 - z^2)(z + i)}$. With $f(z) = z/(9 - z^2)$ the integral is of the form $\displaystyle\int_C \frac{f(z)\,dz}{z - z_0}$ where f is analytic on and inside C and $z_0 = -i$ is interior to C. So by the Cauchy Integral Formula (Theorem 4.50.A),

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = \int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0) = 2\pi i f(-i) = 2\pi i \frac{(-i)}{9 - (-i)^2} = \frac{\pi}{5}.$$

Revised: 1/26/2020