Section 4.51. An Extension of the Cauchy Integral Formula

Note. In the previous section, we saw that the value of an analytic function can be expressed as a certain type of integral. In this section, we state a result that expresses the nth derivative of an analytic function as a certain type of integral.

Theorem 4.51.A. General Cauchy Integral Formula.
Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C then $f^{(n)}(z_0)$ exists and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Note. This is the first time we have seen that an analytic function (that is, a function which has a first derivative on an open set) is infinitely differentiable.

Note. When $n = 0$, Theorem 4.51.A gives the Cauchy Integral Formula (Theorem 4.50.A). We give a proof of the case when $n = 1$ below (in Lemma 4.51.A) and a proof for $n = 2$ is outlined in Exercise 4.52.9 (in Exercise 4.57.9 in the 9th edition of the book). A proof (in an even more general setting) of Theorem 4.51.A for all $n \in \mathbb{N}$ can be found in my online notes for Complex Analysis 1 (MATH 5510) on IV.5. Cauchys Theorem and Integral Formula (see Theorem IV.5.8). Now, we consider some examples in which we use Theorem 4.55.A to evaluate integrals.

Example 4.51.1. Let C be the positively oriented unit circle $|z| = 1$ and $f(z) = e^{2z}$. Then

$$\int_C \frac{e^{2z}\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{\pi i}{3} 8e^{2(0)} = \frac{8\pi i}{3}. \qquad \square$$

Example 4.51.2. Let z_0 be any point interior to a positively oriented simple closed curve C. When $f(z) = 1$ and $n = 0$, we have $\displaystyle\int_C \frac{dz}{(z-z_0)^1} = 2\pi i$ and for $n \in \mathbb{N}$,

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) = 0. \qquad \square$$

Example 4.51.A. (Example 4.55.3 in the 9th edition of the book.) Let $n \in \mathbb{N} \cup \{0\}$ and let $f(z) = (z^2 - 1)^n$. Then the General Cauchy Integral Formula (Theorem 4.51.A) implies

$$\frac{d^n}{dz^n}[(z^2-1)^n] = \frac{n!}{2\pi i} \int_C \frac{(s^2-1)^n\,ds}{(s-z)^{n+1}}$$

where C is any simple closed contour around z. The Legendre polynomials are defined as

$$P_n(z) = \frac{1}{n!2^n} \frac{d^n}{dz^n}[(z^2-1)^n] \text{ where } n \in \mathbb{N} \cup \{0\},$$

so we can write

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n\,ds}{(s-z)^{n+1}}.$$

Since

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{((s-1)(s+1))^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1}$$

then we have

$$P_n(1) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s+1)^n\,ds}{s-1} \text{ for } n \in \mathbb{N} \cup \{0\},$$

and with $f(s) = (s+1)^n$ we know

$$\int_C \frac{f(s)\,ds}{s-z} = 2\pi i f^{(0)}(z) = 2\pi i f(z) \text{ or } \int_C \frac{(s+1)^n}{s-(1)}\,ds = 2\pi i((1)+1)^n = 2^{n+1}\pi i$$

so that

$$P_n(1) = \frac{1}{2^{n+1}\pi i}(2^{n+1}\pi i) = 1 \text{ for } n \in \mathbb{N} \cup \{0\}.$$

In Exercise 4.52.8 (Exercise 4.57.8 in the 9th edition of the book) it is to be shown that $P_n(-1) = (-1)^n$ for $n \in \mathbb{N} \cup \{0\}$. □

Lemma 4.51.A. Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z is any point interior to C then $f'(z)$ exists and

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds.$$

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