Section 4.52. Some Consequences of the Extension

Note. In Section 4.51 we stated:

> **Theorem 4.5.1.A. Extended Cauchy Formula.** Let f be analytic everywhere inside and on a simple closed contour C, parameterized in the positive sense. If z_0 is any point interior to C then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \text{ for } n = 0, 1, 2, \ldots.$$

This is one of the fundamental results in complex analysis! We support this claim by considering some implications of the result in this and the following two sections.

Note. The next amazing result says that if f is analytic at a given point (that is, if the f is differentiable in some neighborhood of the point) then all orders of the derivative of f exist. That is, if f is analytic at z_0 then f is infinitely differentiable at z_0! Notice that this is not at all how real functions behave. For example, $f(x) = \begin{cases} x^2/2 & \text{for } x \geq 0 \\ -x^2/2 & \text{for } x < 0 \end{cases}$ is differentiable on all of $\mathbb{R}$ (we have $f'(z) = |z|$), but it does not even have a second derivative at $x = 0$. You should interpret this as the fact that the condition of differentiability for a function of a complex variable is much more restrictive than the condition of differentiability of a function of a real variable.

Theorem 4.52.1. If a function f is analytic at a given point, then its derivatives of all orders are analytic at that point too.

Corollary 4.52.A. If a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is analytic at a point $z_0 = x_0 + iy_0$ then the component functions u and v have continuous partial derivatives of all orders at the point.

Theorem 4.52.2. Morera's Theorem.
Let f be continuous on a domain D. If $\int_C f(z)\,dz = 0$ for every closed contour C in D, then f is analytic throughout D.

Note. For a simply connected domain D, Morera's Theorem is a converse of Theorem 4.48.A

Theorem 4.52.3. Cauchy's Inequality.
Suppose that function f is analytic inside and on a positively oriented circle C_R centered at z_0 with radius R. If M_R is the maximum value of $|f(z)|$ on C_R, then

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n} \text{ for } n \in \mathbb{N}.$$

Note. We use Cauchy's Inequality in the next section to ultimately give a proof of the Fundamental Theorem of Algebra.

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