Section 4.53. Liouville's Theorem and the Fundamental Theorem of Algebra

Note. In this section, we use Cauchy's Inequality (which is based on the Extended Cauchy Formula) to show that the only bounded entire functions are constant functions (Liouville's Theorem). We then use this result to prove what I view as another fundamental result in complex analysis: The Fundamental Theorem of *Algebra*.

Theorem 4.53.1. Liouville's Theorem.
If a function f is entire and bounded in the whole complex plane, then f is constant throughout the entire complex plane.

Note. Liouville's Theorem is another example of a behavior of a function of a complex variable which is not shared by a function of a real variable. For example, $\sin x$ is a bounded function of a real variable that is "entire" (that is, differentiable on the entire real line, $\mathbb{R}$). Also, $f(x) = 1/(1+x^2)$ is a rational function of a real variable that is bounded and "entire."

Theorem 4.53.2. The Fundamental Theorem of Algebra.
Any complex polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, of degree $n \geq 1$ has at least one zero. That is, there exists at least one point $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Note. In the next set of exercises in the text (namely, Exercise 4.54.9; Exercise 4.59.8 in the 9th edition of the book) it is shown that if P is a polynomial of degree n and $P(z_0) = 0$ then $P(z) = (z - z_0)Q(z)$ where Q is a polynomial of degree $n - 1$. In a more general, algebraic setting, namely a ring of polynomials over a field, this result is called the "Factor Theorem" (see Corollary 23.3 in my online notes for Introduction to Modern Algebra [MATH 4127/5127] on Section IV.23. Factorizations of Polynomials over a Field). Inductively it allows us to factor P as

$$P(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

for some (not necessarily distinct) $z_1, z_2, \ldots, z_n \in \mathbb{C}$. That is, an n degree complex polynomial has n zeros (counting multiplicity).

Note. One might expect that there is an algebraic proof (that is, a proof using group, ring, and field theory exclusively) of the Fundamental Theorem of *Algebra*. However, this is not the case! There are no purely algebraic proofs (see Israel Kleiner's *A History of Abstract Algebra*, Birkhäuser (2007), page 12). There are proofs which are mostly algebraic, but borrow results from analysis. Namely, the facts that (A) a positive real number has a square root and (B) an odd degree polynomial has a real zero. (A) follows from the Axiom of Completeness of $\mathbb{R}$, and (B) follows from the Intermediate Value Theorem which is also based on the Axiom of Completeness. Such a proof is given in graduate level Modern Algebra 1; see my online notes and Theorem V.3.19 in Section V.3.Appendix. The Fundamental Theorem of Algebra.

Note. In fact, the proof of the Fundamental Theorem of Algebra given in John Fraleigh's *A First Course in Abstract Algebra*, 7th edition (Addison-Wesley, 2003) is the proof we have given here based on Liouville's Theorem. See my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Section VI.31. Algebraic Extensions. This raises the philosophical quest: "Why is the Fundamental Theorem of Algebra called the 'Fundamental Theorem of *Algebra*' if there is no algebraic proof?" Perhaps it should be "Liouville's Corollary"!

Note. The above notes reveal a weakness in a purely algebraic approach. The problem is that the real numbers cannot be defined in a purely algebraic way without an appeal to the analytic concept of a continuum. The continuum property of the real numbers is implied by the Axiom of Completeness; without this, there are no square roots of positive reals and no Intermediate Value Theorem... and no Fundamental Theorem of Algebra! One might observe that the Fundamental Theorem of Algebra would be more appropriately perceived as an interesting property of a particular type of entire function and hence a result from the realm of complex analysis. For some history on the Fundamental Theorem see my online supplement from Modern Algebra 1 (MATH 5410) Supplement: The Fundamental Theorem of Algebra—History.

Revised: 4/23/2020