Section 4.54. Maximum Modulus Principle

Note. In this section we present a major result which has many applications. Some of the applications are illustrated in the supplement to this section.

Lemma 4.54.A. Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

Note. We use Lemma 4.54.A to prove the Maximum Modulus Theorem, but first we elevate equation (2) from the proof of Lemma 4.51.A to the status of a theorem itself.

Theorem 4.54.B. Gauss's Mean Value Theorem.
Let f be analytic on and inside the positively oriented circle $|z - z_0| = \rho$ centered at $z_0 \in \mathbb{C}$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta.$$

Note. Recall that, by definition, the average value of a real-valued function of a real variable f on $[a, b]$ is

$$\mathrm{Av}(f) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

It is this sense that the term "mean" is used in Gauss's *Mean* Value Theorem.

Theorem 4.54.C. The Maximum Modulus Theorem.
If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.

Theorem 4.54.D. Maximum Modulus Theorem, Alternative Version.
Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ on R, which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of R and never in the interior.

Theorem 4.54.E. Let f be continuous on a closed bounded region R, and analytic and not constant on the interior of R. For $f(z) = u(x,y)+iv(x,y)$, where $z = x+iy$, function $u(x,y)$ attains its maximum value in R on the boundary of R and not in the interior.

Note. The proof of the following is to be given in Exercise 4.54.3.

Corollary 4.54.F. The Minimum Modulus Theorem.
Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, then $|f(z)|$ has a minimum value m in R and the minimum is attained at some boundary point of R and is never attained at an interior point of R.

Note. Another version of the Maximum Modulus Theorem is the following, a proof of which is given in my online class notes for Complex Analysis (MATH 5510-20) on Section VI.1. The Maximum Principle.

Theorem 4.54.G. Maximum Modulus Theorem for Unbounded Domains (Simplified 1).
Let $r > 0$ and suppose f is analytic for $|z| > r$, continuous on $|z| = r$, bounded by M on $|z| = r$, and $\lim_{|z|\to\infty} |f(z)| \leq M$. Then $|f(z)| \leq M$ on $|z| \geq r$.

Revised: 4/26/2020