Chapter 5. Series

Note. In this chapter we deal with sequences and series of complex numbers, power series, series with both positive and negative exponents ("Laurent series"), integration and differentiation of series, and series representation of analytic functions.

Section 5.55. Convergence of Sequences

Note. In this section we define sequences and limits of sequences of complex numbers. The definitions are almost identical to those seen in Calculus 2.

Definition. An infinite *sequence* of complex numbers is a function $f : \mathbb{N} \to \mathbb{C}$. We denote a sequence as $z_1, z_2, \ldots, z_n, \ldots$, or $\{z_n\}$, where $z_n = f(n)$. The *limit* of sequence $z_1, z_2, \ldots$ is z if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ whenever $n > n_0$. When the limit exists and is z, the sequence is said to *converge* to z, denoted $\lim_{n\to\infty} z_n = z$. If the limit does not exist then the series *diverges*.

Note. The geometric interpretation of the definition of limit of a sequence is that for any $\varepsilon > 0$, the terms of the sequence eventually (that is for $n > n_0$) lie in the open disk of center z and radius ε:

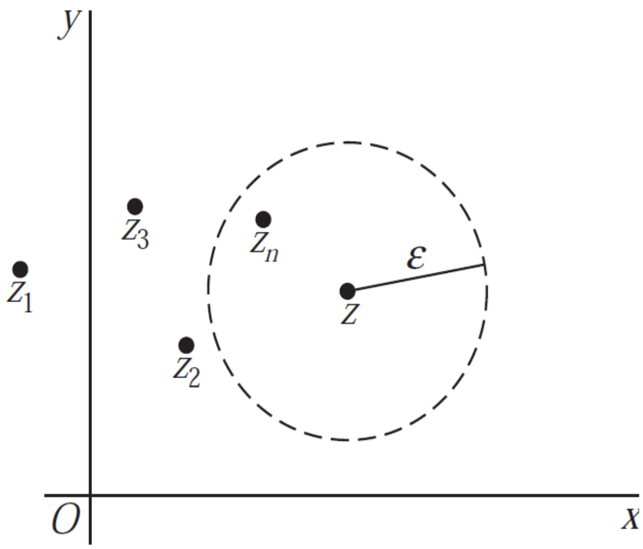

FIGURE 73

Theorem 5.55.A. Suppose that $z_n = x_n + iy_n$ and $z = x + iy$. Then $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Note. Theorem 5.55.A allows us to conclude that $\lim_{n\to\infty}(x_n+iy_n) = \lim_{n\to\infty}(x_n) + i\lim_{n\to\infty}(y_n)$. So if we can write $z_n = x_n + iy_n$, then we can use our knowledge of limits of sequences of real numbers from Calculus 2 (including L'Hopital's Rule) to evaluate complex limits. Brown and Churchill address the examples $z_n = \dfrac{1}{n^3} + i$ and $z_n = -2 + i\dfrac{(-1)^n}{n^2}$.

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