Section 5.56. Convergence of Series

Note. In this section we consider series of complex numbers. We define convergence of series, give a Test for Divergence, and show that absolute convergence of a series implies the convergence of the series.

Definition. An infinite *series* of complex numbers is an infinite sum of the form $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$. This series *converges* if the sequence of partial sums converges and the limit of the partial sums is the *sum* of the series. That is, $\sum_{n=1}^{\infty} z_n = S$ if $S = \lim_{N\to\infty} S_N$ where $S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N$. If $\sum_{n=1}^{\infty} z_n$ does not converge then it *diverges*.

Theorem 5.56.A. Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

Corollary 5.56.1. Test for Divergence

If a series of complex numbers converges, then the nth term converges to zero as n tends to infinity. That is, if z_n does not converge to 0 then $\sum_{n=1}^{\infty} z_n$ diverges.

Corollary 5.56.A. If the series $\sum_{n=1}^{\infty} z_n$ converges, then the sequence $\{z_n\} = z_1, z_2, \ldots$ is a bounded sequence. That is, there is positive $M \in \mathbb{R}$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$.

Note. A proof of Corollary 5.56.A is to be given in Exercise 5.56.9.

Definition. A series of complex numbers $\sum_{n=1}^{\infty} z_n$ is *absolutely convergent* if the series of real numbers $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Note. As with series of real numbers, absolute convergence implies convergence as we now show.

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Definition. Let $\sum_{n=1}^{\infty} z_n$ be a series with sum S and Nth partial sum $S_N = \sum_{n=1}^{N} z_n$. The *remainder* ρ_N after N terms Is $\rho_N = S - S_N$.

Note 5.56.A. Notice that series $\sum_{n=1}^{\infty} z_n$ converges to S if and only if sequence $\{\rho_N\} = \rho_1, \rho_2, \ldots$ converges to 0.

Definition. A *power series* centered at z_0 is a series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ where z is a variable and the a_n are *coefficients*. The sums, partial sum, and remainder of a power series are denoted $S(z)$, $S_N(z)$, and $\rho_N(z)$, respectively.

Note. We explore power series in the next section. Notice that when $n = 0$ and $z = z_0$, the first term in the power series seems to involve 0^0. However, this is simply a shortcoming of the power series notation. When $z = z_0$ we interpret the first term of the power series as a_0.

Example. The most important types of series are geometric series. This is because we can actually find the sum of a geometric series and much of the theory of power series are based on geometric series. Consider the geometric series $\sum_{n=1}^{\infty} z^n$ where $|z| < 1$. We claim:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ whenever } |z| < 1.$$

To establish this, we notice from Exercise 1.8.9 (Exercise 1.9.9 in the 9th edition of the book) that

$$1 + z + z^2 + z^3 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \text{ for } z \neq 1.$$

Notice that the Nth partial sum for the geometric series is

$$S_N(z) = \sum_{n=0}^{N-1} z^n = 1 + z + z^2 + \cdots + z^{N-1}$$

and so $S_N(z) = \dfrac{1 - z^N}{1 - z}$. With $S(z) = \dfrac{1}{1-z}$ we have the remainder $\rho_N(z) = S(z) - S_N(z) = \dfrac{z^N}{1-z}$. So $|\rho_N(z)| = \dfrac{|z|^N}{|1-z|}$ and for $|z| < 1$ we have

$$\lim_{n\to\infty} |\rho_N(z)| = \lim_{n\to\infty} \frac{|z|^N}{|1-z|} = \frac{\lim_{n\to\infty} |z|^N}{\lim_{n\to\infty} |1-z|} = \frac{0}{|1-z|} = 0.$$

So, for $|z| < 1$, $\displaystyle\sum_{n=0}^{\infty} z^n = S(z) = \frac{1}{1-z}$.

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