Section 5.57. Taylor Series

Note. In this section we state a theorem claiming that a function analytic in an open disk (that is, differentiable at each point of the disk) is a function with a power series representation in that disk. The proof of this result is given in the next section.

Theorem 5.57.A. Taylor's Theorem. Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$ (that is, $f'(z)$ is defined for each $|z - z_0| < R_0$), centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $|z - z_0| < R_0$ where $a_n = f^{(n)}(z_0)/n!$ for $n = 0, 1, 2, \ldots$. that is, the series converges to $f(z)$ for each z in the stated disk.

Definition. The series for analytic function f in Theorem 5.57.A is the *Taylor series* for f centered at z_0. If $z_0 = 0$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R_0$, then the series is called a *Maclaurin series*.

Note 5.57.A. Theorem 5.57.A completes the conversation started in Section 2.24. As we stated there, in the real setting the term "analytic" refers to a function with a power series representation. We now see that our definition of analytic as differentiable in a neighborhood implies this existence of a power series representation. The uniqueness of this series representation is shown in Theorem 5.66.1.

Note. The result given in Theorem 5.57.A is truly remarkable. In the real setting a function can be differentiable but not twice differentiable. Consider the function

$$f(x) = \begin{cases} x^2 & \text{for } x \in [0, \infty) \\ -x^2 & \text{for } x \in (-\infty, 0). \end{cases}$$

Then $f'(x) = |x|$ and so $f''(0)$ does not exist. In general, a function of a real variable can be n times differentiable by not $n+1$ times differentiable. In fact, a function of a real variable can be infinitely differentiable but still not have a power series representation (for details, see my online notes from Analysis 2 [MATH 4227/5227] on 8.3. Taylor Series). However, from Theorem 5.57.A we see that if a function of a complex function is differentiable in an open neighborhood then it is infinitely differentiable and has a power series representation. For an in-depth discussion of the behavior of real functions versus complex functions in terms of differentiability and series representations, see my online notes for Complex Analysis 1 (MATH 5510) on A Primer on Lipschitz Functions.

Revised: 1/29/2020