Section 5.58. Proof of Taylor's Theorem

Note. In this section we give a proof of Taylor's Theorem that an analytic function on a disk has a power series representation on that disk:

> **Theorem 5.57.A. Taylor's Theorem.** Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$ (that is, $f'(z)$ is defined for each $|z - z_0| < R_0$), centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $|z - z_0| < R_0$ where $a_n = f^{(n)}(z_0)/n!$ for $n = 0, 1, 2, \ldots$. that is, the series converges to $f(z)$ for each z in the stated disk.

Note. Notice that the proof of Taylors Theorem depends heavily on properties of complex *integrals*. In Calculus 2 series representations are built up by considering progressively higher orders of derivatives (see my Calculus 3 notes on 10.8. Taylor and Maclaurin Series). The proof also depends on "your favorite" type of series, geometric series (where the last Example from Section 56 is referenced). Geometric series are nice because (1) we can easily tell whether or not a geometric series converges, (2) when a geometric series does converge, we can determine the sum of the series (a rare ability when dealing with series: in Calculus 2 you are often asked about whether or not a series converges, but very rarely asked about the sum of the series), and (3) geometric series play an important theoretical role in our other knowledge about series.

Revised: 1/29/2020