Section 5.59. Examples

Note. We now apply Taylor's Theorem (Theorem 5.57.A) to find series representations for several functions. In each example, we must be aware of parameter R_0.

Example 5.59.1. The function $f(z) = e^z$ is entire (since $f'(z) = e^z$ for all $z \in \mathbb{C}$ by Exercise 2.22.A) so by Taylor's Theorem, $f(z)$ has a Maclaurin series $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ valid for all $z \in \mathbb{C}$ (that is, $R_0 = \infty$). Here, $f^{(n)}(z) = e^z$ for $n \in \mathbb{N} \cup \{0\}$ and so $f^{(n)}(0) = 1$ for all n. So

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \text{ for } |z| < \infty.$$

We can find a series for the entire function $z^2 e^{3z}$ by replacing z with $3z$ in the above series (and noticing that $|3z| < \infty$ is equivalent to $|z| < \infty$) to get first that

$$e^{3z} = \sum_{n=0}^{\infty} \frac{1}{n!}(3z)^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^n \text{ for } |z| < \infty.$$

Next we multiply both sides by z^2 and distribute on the right-hand side (this can by justified pointwise by Exercise 5.56.7) to get

$$z^2 e^{3z} = z^2 \sum_{n=0}^{\infty} \frac{3^n}{n!} z^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \text{ for } |z| < \infty.$$

We can similarly find a series for e^{-z^2} as

$$e^{-z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{2n} \text{ for } |z| < \infty.$$

With $z = x$ real, we have $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for $|x| < \infty$, which is a useful series in statistics (it is related to the normal distribution and can be used to calculate the numerical values in a Z-table).

Example 5.59.2. In Section 3.34, "Trigonometric Functions," we defined $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$. Since we now have a Maclaurin series for e^z, we can present such a series for $\sin z$:

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{1}{n!}(iz)^n - \sum_{n=0}^{\infty} \frac{1}{n!}(-iz)^n\right) \text{ for } |z| < \infty \\
&= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n i^n}{n!} z^n\right) \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(1 - (-1)^n) i^n}{n!} z^n \text{ (this can be justified pointwise by Exercise 5.56.8)} \\
&= \frac{1}{2i} \sum_{n=0, n \text{ odd}}^{\infty} \frac{(1 - (-1)^n) i^n}{n!} z^n \text{ since } (1) - (-1)^n = 0 \text{ for } n \text{ even} \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(1 - (-1)^{2n+1}) i^{2n+1}}{(2n+1)!} z^{2n+1} \text{ replacing odd } n \text{ above} \\
&\quad \text{with } 2n+1 \text{ here} \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{2(-1)^n i}{(2n+1)!} z^{2n+1} \text{ since } 1 - (-1)^{2n+1} = 2 \text{ and } i^{2n+1} = (i^2)^n i = (-1)^n i \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \text{ for } |z| < \infty.
\end{aligned}$$

So

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \text{for } |z| < \infty.$$

In Theorem 5.65.2 we'll see that a power series can be differentiated term-by-term so that

$$\begin{aligned}
\cos z &= \frac{d}{dz}[\sin z] = \frac{d}{dz}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right] \\
&= \sum_{n=0}^{\infty}\frac{d}{dz}\left[\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right] \text{ by Theorem 5.65.2} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+1)z^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n} \text{ for } |z| < \infty.
\end{aligned}$$

Example 5.59.4. Consider $f(z) = \dfrac{1}{1-z} = (1-z)^{-1}$. We have

$$f^{(n)}(z) = n!(1-z)^{-(n+1)} = \frac{n!}{(1-z)^{n+1}} \text{ and } f^{(n)}(0) = n! \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Now $f(z)$ is not defined at $z = 1$ so that the Maclaurin series for $f(z)$ can have radius of convergence R_0 at most 1. The Maclaurin series is

$$\frac{1}{1-z} = f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n = \sum_{n=0}^{\infty}\frac{n!}{n!}z^n = \sum_{n=0}^{\infty}z^n,$$

and as we saw in an example from Section 5.56, "Convergence of Series," this series in fact converges for $|z| < R_0 = 1$. So we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty}z^n \text{ for } |z| < 1.$$

If we replace z with $1-z$ then we get the Taylor series

$$\frac{1}{z} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n \text{ for } |z-1| < 1.$$

Example 5.59.5. We seek a series representation of the rational function

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right).$$

First, we use Example 5.59.4 to get $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ for $|z| < 1$, and replacing z with $-z^2$ we see that

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n} \text{ for } |z| < 1$$

and so (since $1/z^3$ is not defined at $z = 0$):

$$\begin{aligned}
f(z) &= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right) = \frac{1}{z^3}\left(2 - \sum_{n=0}^{\infty}(-1)^n z^{2n}\right) \text{ for } 0 < |z| < 1 \\
&= \frac{2}{z^3} - \frac{1}{z^3}\sum_{n=0}^{\infty}(-1)^n z^{2n} \\
&= \frac{2}{z^3} - \sum_{n=0}^{\infty}(-1)^n z^{2n-3} \text{ (this can be justified pointwise by Exercise 5.56.7)} \\
&= \frac{2}{z^3} - \frac{1}{z^3} + \frac{1}{z} - \sum_{n=2}^{\infty}(-1)^n z^{2n-3} \\
&= \frac{1}{z^3} + \frac{1}{z} - \sum_{n=2}^{\infty}(-1)^n z^{2n-3} \text{ for } 0 < |z| < 1.
\end{aligned}$$

This is a series representation of f, but it is not a Taylor series since it involves some negative powers of z. We'll see in the next section that such a series is called a "Laurent series."

Revised: 2/2/2020