Section 5.60. Laurent Series

Note. In this section we consider a function f analytic "near" but not at $z - a$. We cannot associate a power series about $z = a$ with f since it is not analytic at a but we can sort of isolate the "bad" part of f and produce a different kind of series for f. The theory is based on the following theorem which we prove in the next section.

Theorem 60.1. Laurent's Theorem.

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain (see Figure 76). Then at each point in the domain, $f(z)$ has the series representation

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \text{ for } R_1 < |z - z_0| < R_2$$

where $$(2) \qquad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \; n = 0, 1, 2, \ldots$$

and $$(3) \qquad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}, \; n = 1, 2, 3, \ldots.$$

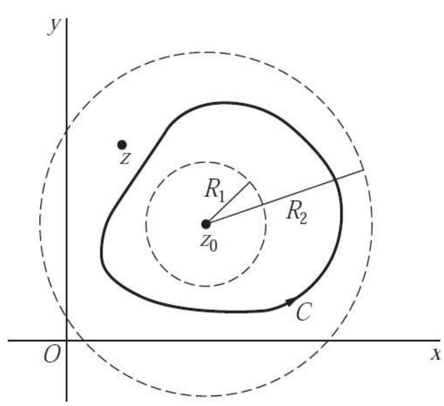

FIGURE 76

Note 5.60.A. We can simplify Theorem 60.1 by expressing $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where $R_1 < |z - z_0| < R_2$ and

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \text{ for } n \in \mathbb{Z}.$$

These representations of f are *Laurent series* for f. We have not defined what the "double series" given here means, but we just interpret it as notation for the sum of the series given in Equation (1).

Note. If f is analytic in $|z - z_0| < R_2$ then for $n \in \mathbb{Z}$ with $n < 0$,

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} = \frac{1}{2\pi i} \int_C f(z)(z - z_o)^{-n+1}\,dz = 0$$

by Theorem 4.48.A, since $f(z)(z - z_0)^{-n-1}$ is analytic in $|z - z_0| < R_2$. So in this case, the Laurent series reduces to a power series. In fact, by the Extension of The Cauchy Integral Theorem (Theorem 51.1) we have

$$\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \text{ for } n \in \mathbb{Z} \text{ and } n \geq 0,$$

so $c_n = f^{(n)}(z_0)/n!$ and the Laurent series actually reduces to the Taylor series.

Note. A Laurent series can be produced if f is analytic on $0 < |z - z_0| < R_2$ (that is, when $R_1 = 0$). In fact, this is the case in most of our examples (see Section 5.62).

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