Section 5.61. Proof of Laurent's Theorem

Note. We now give a moderately lengthy proof of Laurent's Theorem (Theorem 5.60.1).

Proof of Laurent's Theorem. Let f be analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C be any positively oriented simple closed contour around z_0 and lying in that domain. Form a closed annular region $r_1 < |z| < r_2$ that is contained in the domain $R_1 < |z| < R_2$ and whose interior contains both the point z and the contour C (see Figure 77).

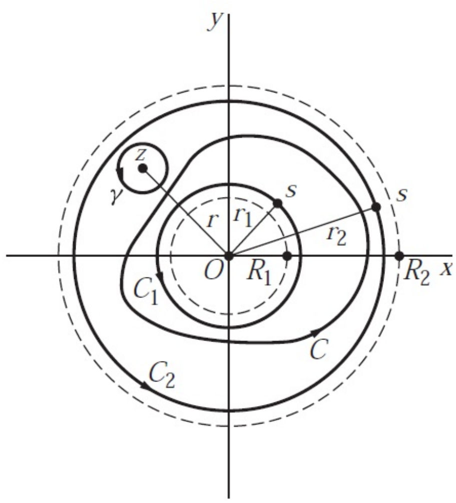

FIGURE 77

Let C_1 and C_2 be the circles $|z| = r_1$ and $|z| = r_2$, respectively, and give each a positive orientation.

First we consider the case $z_0 = 0$. Let z be a point in $r_1 < |z| < r_2$ and let γ be a positively oriented circle with center z and small enough to be contained in the annular region $r_1 < |z| < r_2$ (see Figure 77). By the Cauchy-Goursat Theorem for analytic functions around oriented boundaries of multiply connected domains

(Theorem 4.49.A; see Figure 60) we have

$$\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)\,ds}{s-z} - \int_{\gamma} \frac{f(s)\,ds}{s-z} = 0.$$

By the Cauchy Integral Formula (Theorem 4.50.A) we have $\int_{\gamma} \frac{f(s)\,ds}{s-z} = 2\pi i f(z)$, so that

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)\,ds}{z-s} + \frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,ds}{z-s}. \qquad (*)$$

Now

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N \frac{1}{(s-z)s^N} \qquad (**)$$

as shown in the proof of Taylor's Theorem (see Section 5.58) and interchanging s and z gives

$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \frac{1}{z^{n+1}} + \frac{1}{z^N}\frac{s^N}{z-s} = \sum_{n=1}^{N} \frac{1}{s^{-n+1}} \frac{1}{z^n} + \frac{1}{z^N}\frac{s^N}{z-s} \text{ (reindexing)} \qquad (***)$$

From $(**)$ we have

$$\frac{1}{2\pi i}\frac{f(s)}{s-z} = \frac{1}{2\pi i}\sum_{n=0}^{N-1} \frac{f(s)}{s^{n+1}} z^n + \frac{f(s)}{(s-z)s^N} z^N$$

and so

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{s-z}\,ds = \frac{1}{2\pi i}\sum_{n=0}^{N-1}\left(\int_{C_2} \frac{f(s)\,ds}{s^{n+1}}\right) z^n + \frac{1}{2\pi i}\left(\int_{C_2} \frac{f(s)\,ds}{(s-z)s^N}\right) z^N.$$

Similarly, from $(***)$

$$\frac{1}{2\pi i}\frac{f(s)}{z-s} = \frac{1}{2\pi i}\sum_{n=1}^{N} \frac{f(s)\,ds}{s^{-n+1}}\frac{1}{z^n} + \frac{1}{2\pi i}\frac{s^N f(s)}{z-s}\frac{1}{z^N}$$

and so

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(s)\,ds}{z-s} = \frac{1}{2\pi i}\sum_{n=1}^{N}\left(\int_{C_1} \frac{f(s)\,ds}{s^{-n+1}}\right)\frac{1}{z^n} + \frac{1}{2\pi i}\left(\int_{C_1} \frac{s^N f(s)\,ds}{z-s}\right)\frac{1}{z^N}.$$

So from $(*)$,

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{N-1}\left(\int_{C_2}\frac{f(s)\,ds}{s^{n+1}}\right)z^n + \frac{1}{2\pi i}\left(\int_{C_2}\frac{f(s)\,ds}{(s-z)s^N}\right)z^N$$

$$+\frac{1}{2\pi i}\sum_{n=1}^{N}\left(\int_{C_1}\frac{f(s)\,ds}{s^{-n+1}}\right)\frac{1}{z^n} + \frac{1}{2\pi i}\left(\int_{C_1}\frac{s^N f(s)\,ds}{z-s}\right)\frac{1}{z^N}$$

$$= \sum_{n=0}^{N-1} a_n z^n + \rho_N(z) + \sum_{n=1}^{N}\frac{b_n}{z^n} + \sigma_N(z)$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(s),ds}{s^{n+1}} \text{ for } n = 0,1,\ldots,N-1$$

$$b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{s^{-n+1}} \text{ for } n = 1,2,\ldots,N,$$

$$\rho_N(z) = \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)\,ds}{(s-z)s^N}, \text{ and } \sigma_N(z) = \frac{1}{2\pi i z^N}\int_{C_1}\frac{s^N f(s)\,ds}{z-s}.$$

Next, for the given z let $|z| = r$ where $r_1 < r < r_2$, and let M denote the maximum value of $|f(s)|$ on C_1 and C_2 (which exists since $|f(s)|$ is continuous and $C_1 \cup C_2$ is compact). If s is any point on C_2 then $|s - z| \geq r_2 - r$, and if s is any point on C_1 then $|z - s| \geq r - r_1$. So

$$\begin{aligned}|\rho_N(z)| &= \left|\frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)\,ds}{(s-z)z^N}\right| \\ &\leq \frac{r^N}{2\pi}\frac{M2\pi r_2}{(r_2-r)r_2^N} \text{ by Theorem 4.43.A} \\ &= \frac{Mr_2}{r_2-r}\left(\frac{r}{r_2}\right)^N\end{aligned}$$

and

$$|\sigma_N(z)| = \left|\frac{1}{2\pi i z^N}\int_{C_1}\frac{s^N f(s)\,ds}{z-s}\right|$$

$$\begin{aligned} &\le \frac{1}{2\pi r^N}\frac{Mr_1^N 2\pi r_1}{r-r_1} \text{ by Theorem 4.43.A} \\ &= \frac{Mr_1}{r-r_1}\left(\frac{r_1}{r}\right)^N. \end{aligned}$$

Since $r/r_2 < 1$ and $r_1/r < 1$, then as $N \to \infty$ we have $\rho_N(z) \to 0$ and $\sigma_N(z) \to 0$. Therefore,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

where a_n and b_n are as required. Since z is an arbitrary point in $r_1 < |z| < r_2$, then this holds for all such z and the theorem holds for $z_0 = 0$. (Notice that we need to replace s with z so that a_n and b_n are in the form stated in the theorem.)

We now consider the case $z_0 \neq 0$. With f as required, define $g(z) = f(z + z_0)$. Since f is analytic in $R_1 < |z-z_0| < R_2$ then $g(z)$ is analytic in $R_1 < |(z+z_0)-z_0| = |z| < R_2$. With simple closed contour C given as $z = z(t)$ where $a \le t \le b$, let γ denote the simple closed contour $z = z(t) - z_0$ where $a \le t \le b$. Then γ lies in $R_1 < |z| < R_2$. So by the first part of the proof, $g(z)$ as a Laurent series representation centered at $z_0 = 0$:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \text{ for } R_1 < |z| < R_2 \text{ where}$$

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)\,dz}{z^{n+1}} \text{ for } n = 0, 1, 2, \ldots \text{ and } b_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)\,dz}{z^{-n+1}} \text{ for } n = 1, 2, \ldots.$$

That is, since $f(z) = g(z - z_0)$,

$$f(z) = g(z-z_0) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \text{ for } R_1 < |z-z_0| < R_2 \text{ where}$$

$$a_n = \frac{1}{2\pi i}\int_{z(t)-z_0} \frac{g(z)\,dz}{z^{n+1}} = \frac{1}{2\pi i}\int_a^b \frac{f(z(t))z'(t)\,dt}{(z(t)-z_0)^{n+1}}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \text{ for } n = 0, 1, 2, \ldots \text{ and}$$

$$b_n = \frac{1}{2\pi i} \int_{z(t)-z_0} \frac{g(z)\,dz}{z^{-n+1}} = \frac{1}{2\pi i} \int_a^b \frac{f(z(t))z'(t)\,dt}{(z(t)-z_0)^{-n+1}}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \text{ for } n = 1, 2, \ldots.$$

Therefore, the claim holds for general z_0, as claimed. ∎

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