Section 5.62. Examples

Note. We now find Laurent series for several functions. In each example, we are careful to give a set on which the series is valid. However, we do not compute coefficients using integrals as stated in Laurent's Theorem (Theorem 5.60.1).

Example 5.62.1. Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $|z| < \infty$, then replacing z with $1/z$ we have

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} \text{ for } 0 < |z| < \infty.$$

Based on Laurent's Theorem (Theorem 5.60.1), we see that $b_1 = \frac{1}{2\pi i}\int_C e^{1/z}\,dz$ where C is any positively oreinted simple closed contour around $z_0 = 0$. Since $b_1 = 1$ here, then $\int_C e^{1/z}\,dz = 2\pi i$. So if we have a Laurent series for $f(z)$, then we can use it and Laurent's Theorem to evaluate certain integrals. This is explained in more detail in Chapters 6 and 7 ("Residues and Poles" and "Applications of Residues," respectively).

Example 5.62.2. The function $f(z) = \frac{1}{(z-i)^2}$ is already in the form of a Laurent series where $z_0 = i$. With

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n(z-i)^n \text{ where } 0 < |z-i| < \infty$$

we have $c_{-2} = 1$ and $c_n = 0$ for $n \in \mathbb{Z} \setminus \{-2\}$. By Laurent's Theorem (see Note 5.60.A)

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \text{ for } n \in \mathbb{Z}$$

and C any positively oriented simple closed contour around $z_0 = i$ lying in that domain $0 < |z - i| < \infty$. Therefore

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{for } n \in \mathbb{Z} \setminus \{-2\} \\ 2\pi i & \text{for } n = -2. \end{cases}$$

Examples 5.62.3 and 5.62.4. Consider the function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

This has two singular points, $z = 1$ and $z = 2$. So $f(z)$ is analytic in $\mathbb{C} \setminus \{1, 2\}$. In particular, f is analytic in the domains $|z| < 1$, $1 < |z| < 2$, and $2 < |z| < \infty$, which we denote D_1, D_2, and D_3 respectively (see Figure 78).

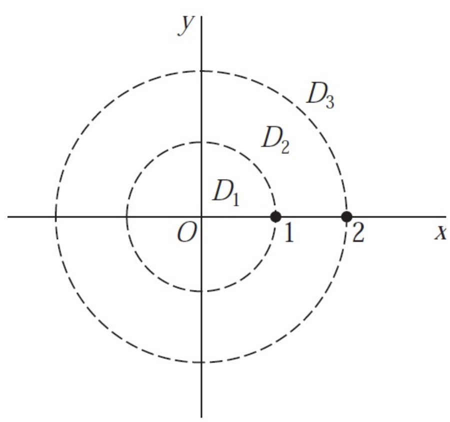

Figure 78

As seen above, we have $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. Replacing z with $z/2$ gives

$$\frac{1}{1 - z/2} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{2^n} \text{ for } |z| < 2.$$

So in domain D_1 we have

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1-z} + \frac{1}{2-z} = -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-z/2}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - 1\right) z^n \text{ for } |z| < 1.$$

So $f(z)$ actually has a Taylor series representation on D_1.

Next, we look for a series representation on D_2. We need to modify our version of f so that we can get a series representation valid outside of $|z| \leq 1$. So we write

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{2}\frac{1}{1-z/2}.$$

Replacing z with $1/z$ in the series for $\dfrac{1}{1-z}$ we get

$$\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \text{ for } |1/z| < 1 \text{ (or equivalently } |z| > 1).$$

So in domain D_2 we have

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{2}\frac{1}{1-z/2} \\ &= \frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} \text{ for } 1 < |z| < 2 \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \text{ for } 1 < |z| < 2. \end{aligned}$$

Finally, we look for a series representation on D_3. This time we write

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-1/z} - \frac{1}{z}\frac{1}{1-2/z}.$$

From above,

$$\frac{1}{z}\frac{1}{1-1/z} = \frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \text{ for } |z| > 1.$$

Similarly, replacing z with $2/z$ in the series for $\dfrac{1}{1-z}$ we get

$$\frac{1}{1-2/z} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{z^n} \text{ for } \left|\frac{2}{z}\right| < 1 \text{ (or equivalently } |z| > 2).$$

So in domain D_3 we have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \text{ for } |z| > 2.$$

So in these examples we see that the same function may have different Laurent series centered at z_0 which are valid in different regions.

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