Section 5.63. Absolute and Uniform Convergence of Power Series

Note. In this section, we prove two theorems concerning the convergence of power series. In the following three sections (Sections 64–66) we consider additional properties of series. These sections include theoretical information and not computational information.

Note. Recall (see Section 5.56, "Convergence of Series") that a series of complex numbers $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if the sequence of real numbers $\sum_{n=0}^{\infty} |z_n|$ is convergent.

Theorem 5.63.1. If a power series $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z = z_1$ (where $z_1 \neq z_0$), then the power series is absolutely convergent at each point z in the disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$ (see Figure 79).

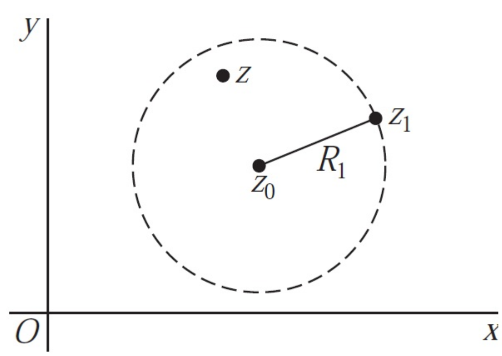

FIGURE 79

Note. Theorem 5.63.1 implies that a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely on a disk centered at z_0 (possibly of infinite radius). So the ideas of an interval of convergence and radius of convergence from calculus and real analysis can be extended to the complex setting.

Definition. The greatest circle centered at z_0 (possibly with the "circle" as all of $\mathbb{C}$) such that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at each point inside the circle is the *circle of convergence* of the series. The radius of the circle of convergence is the *radius of convergence* of the series (which could possibly be infinite).

Note. The power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ may or may not converge for points *on* the circle of convergence (though it will converge absolutely *inside* the circle of convergence by Theorem 5.63.1).

Note. We now extend the idea of a uniform limit of a sequence $\{f_n(x)\}$ of real functions (see my online notes for Analysis 2 on 8.1. Sequences of Functions) to the uniform convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (which is, by definition, the limit of the sequence of partial sums $S_N(z) = \sum_{n=0}^{N-1} a_n(z-z_0)^n$).

Definition. Let $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series centered at z_0, and suppose the power series has circle of convergence $|z - z_0| = R$ (possibly all of $\mathbb{C}$). Define the Nth partial sum $S_N(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n$ and the remainder function $\rho_N(z) = S(z) - S_N(z)$. The sequence of functions $\{S_N(z)\}_{N=1}^{\infty}$ *converges uniformly* to $S(z)$ on some region in the circle of convergence if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $N > N_\varepsilon$ we have $|\rho_N(z)| < \varepsilon$ for all z in the region.

Theorem 5.63.2. If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, then the series is uniformly convergent in every closed disk $|z - z_0| < R_1$ where $0 \leq R_1 < R$.

Revised: 2/10/2020