Section 5.64. Continuity of Sums of Power Series

Note. We now use the uniform convergence introduced in Section 5.63, "Absolute and Uniform Convergence of Power Series," to show that a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, represents a continuous function inside its circle of convergence. You might think that there is an easy way to do this since we know by Theorem 2.19.A that a differential function is continuous, however we have not shown yet that a power series represents a differentiable function (we will do so in the next section).

Theorem 5.64.1. A power series $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z-z_0| = R$.

Note 5.64.A. We can modify Theorems 5.63.1, 5.63.2, and 5.64.1 to apply to Laurent series of the type $\displaystyle\sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}$. If this series converges at a point z_1 (where $z_1 \neq z_0$) then the series $\sum_{n=1}^{\infty} b_n w^n$ converges absolutely to a continuous function (by Theorem 5.64.1) when $|w| < \dfrac{1}{|z_1 - z_0|}$. Now

$$\frac{1}{|z-z_0|} = |w| < \frac{1}{|z_1 - z_0|}$$

implies $|z_1 - z_0| < |z - z_0|$, so Laurent series $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges absolutely (by Theorem 5.63.1) to a continuous function in the domain $|z - z_0| > R_1$ where $R_1 = |z_1 - z_0|$. If a more general Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

is valid in annulus $R_1 < |z - z_0| < R_2$, then both series on the right converge uniformly (by Theorem 5.63.2 and the observation above) in any closed annulus which is centered at z_0 and interior to the annulus $R_1 < |z - z_0| < R_2$.

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