Section 5.65. Integration and Differentiation of Power Series

Note. If a function can be written as a power series then we can numerically approximate function values. In this section we show that integration and differentiation can be performed term-by-term. Since powers of $(z - z_0)^n$ are easy to integrate then it is easy to integrate power series (and as you see in Calculus 2, without using series many functions are hard to integrate, such as $\sin x^2$ and e^{-x^2}).

Theorem 5.65.1. Let C denote any contour interior to the circle of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term-by-term over C; that is

$$\int_C g(z)S(z)\,dz = \int_C g(z)\sum_{n=0}^{\infty} a_n(z - z_0)^n\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\,dz.$$

Note 5.65.A. With $g(z) = 1$, Theorem 5.65.1 implies that, for C in the circle of convergence of the power series,

$$\int_C \sum_{n=0}^{\infty} a_n(z - z_0)^n\,dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n\,dz.$$

Corollary 5.65.1. The power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is analytic at each point z interior to the circle of convergence of the series.

Note 5.65.B. By Taylor's Theorem (Theorem 5.57.A) we know that an analytic function has a power series representation. By Corollary 5.65.1, we see that a power series represents an analytic function. Therefore, **a function is analytic on $|z - z_0| < R$ (where R could be ∞) if and only if it has a power series representation on $|z - z_0| < R$.**

Example 5.65.1. Define $f(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0 \\ 1 & \text{when} z = 0. \end{cases}$ The Maclaurin series $e^z - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n!}$ is valid for all $z \in \mathbb{C}$ and so

$$\frac{e^z - 1}{z} = \frac{1}{z}\sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots \text{ for } z \neq 0$$

and this Maclaurin series represents $f(z)$ for $z \neq 0$. But the series has the value 1 at $z = 0$ and so we can state that $f(z) = \sum_{n=1}^{\infty} z^n/n!$ for all $z \in \mathbb{C}$. So by Corollary 5.65.1, $f(z)$ is analytic in the entire complex plane. In particular,

$$\begin{aligned} \lim_{z\to 0} \frac{e^z - 1}{z} &= \lim_{z\to 0} f(z) \text{ since "}\lim_{z\to 0}\text{" implies that } z \neq 0 \\ &\quad \text{ and } \tfrac{e^z-1}{z} = f(z) \text{ for } z \neq 0 \\ &= f(0) = 1. \end{aligned}$$

Theorem 5.65.2. The power series $S(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z - z_0)^n$ can be differentiated term-by-term in its circle of convergence. That is, at each point z interior to the circle of convergence of that series, we have $S'(z) = \displaystyle\sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$.

Example 5.65.2. In example 5.59.4 we saw that $\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$ for $|z-1| < 1$. So by Theorem 5.65.2, we can differentiate to get $\frac{-1}{z^2} = \sum_{n=1}^{\infty}(-1)^n n(z-1)^{n-1}$ for $|z-1| < 1$, or

$$\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(z+1)^n \text{ for } |z-1| < 1. \qquad \square$$

Revised: 2/10/2020