Section 5.67. Multiplication and Division of Power Series

Note. We know by the Product Rule and the Quotient Rule (Theorem 2.20.B) that a product or quotient of analytic functions is analytic. In this section we show how to find the power series for a product of analytic functions with given power series (about point z_0), and illustrate by two example how we can use long division to find some coefficients of the power series of a quotient of power series.

Note. Suppose that the power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

converge within some circle $|z-z_0| = R$. Let the power series of the product $f(z)g(z)$ be $\sum_{n=0}^{\infty} c_n(z-z_0)$. By repeated use of the product rule we can show Leibniz's Rule (see Exercise 5.67.6, or Exercise 5.73.7 in the 9th edition of the book) for the nth derivative of a product

$$(f(z)g(z))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z) \text{ for } n \in \mathbb{N}.$$

So the nth coefficient in the power series for $f(z)g(z)$ is

$$c_n = \frac{(f(z)g(z))^{(n)}|_{z=z_0}}{n!} = \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)}(z_0)g^{(n-k)}(z_0)$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}\frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}.$$

So

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n \text{ for } |z-z_0| < R.$$

Definition. The *Cauchy product* of power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ and } \sum_{n=0}^{\infty} b_n(z-z_0)^n \text{ convergent for } |z-z_0|<R,$$

is the power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ where $c_n = \sum_{n=0}^{n} a_k b_{n-k}$.

Example 5.67.1. Consider the function $e^z/(1+z)$. It has a singular point at $z=-1$ and we can find Maclaurin series $e^z = \sum_{n=0}^{\infty} z^n/z!$ (valid for all $z \in \mathbb{C}$; see Example 5.59.1) and

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-1)^n z^n \text{ valid for } |z|<1$$

(see Example 5.59.4). So multiplying term-by-term we have

$$\frac{e^z}{1+z} = \left(1+z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\cdots\right)(1-z+z^2-z^3+\cdots)$$

$$= (1)(1)+((1)-(1))z+\left((1)(1)+(1)(-1)+\left(\frac{1}{2}\right)(1)\right)z^2$$

$$+\left((1)(-1)+(1)(1)+\left(\frac{1}{2}\right)(-1)+\left(\frac{1}{6}\right)(1)\right)z^3+\cdots$$

$$= 1+\frac{1}{2}z^2-\frac{1}{3}z^3+\cdots \text{ for } |z|<1.$$

Note. With $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ for $|z-z_0|<R$ and $g(z) \neq 0$ for $|z-z_0|<R$, the quotient $f(z)g(z)$ is analytic and so

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n(z-z_0)^n \text{ for } |z-z_0|<R$$

where the coefficients can be found by differentiating $f(z)/g(z)$ successively and evaluating the derivatives at $z=z_0$. We now give an example where we find coefficients using long division instead.

Example 5.67.2. Consider the function $\dfrac{1}{z^2 \sinh z}$. The zeros of $\sinh z$ are $z = n\pi i$ where $n \in \mathbb{Z}$ (see Note 3.35.C). We have by Example 5.59.3 that

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \text{ for } |z| < 1,$$

so

$$\begin{aligned} \frac{1}{z^2 \sinh z} &= \frac{1}{z^2(z + z^3/3! + z^5/5! + \cdots)} \text{ for } 0 < |z| < \pi \\ &= \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right) \text{ for } 0 < |z| < \pi. \end{aligned}$$

So we perform long division to find the first few terms of the Laurent series for $\dfrac{1}{z^2 \sinh z}$:

$$\begin{array}{r|llll} & 1 & -\frac{1}{3!}z^2 & +\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^4 & +\cdots \\ \hline 1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots & 1 & & & \\ & 1 & +\frac{1}{3!}z^2 & +\frac{1}{5!}z^4 & +\cdots \\ \hline & & -\frac{1}{3!}z^2 & -\frac{1}{5!}z^4 & +\cdots \\ & & -\frac{1}{3!}z^2 & -\frac{1}{(3!)^2}z^4 & +\cdots \\ \hline & & & \vdots & \end{array}$$

That is,

$$\begin{aligned} \frac{1}{1 + z^2/3! + z^4/5! + \cdots} &= 1 - \frac{1}{3!}z^2 + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^4 + \cdots \\ &= 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots \text{ for } |z| < \pi. \end{aligned}$$

Hence

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{7}{360}z + \cdots \text{ for } 0 < |z| < \pi. \qquad \square$$