Chapter 6. Residues and Poles

Note. Now that we have studies Laurent series, we use them to evaluate integrals. This requires us to first classify singularities.

Section 6.68. Isolated Singular Points

Note. Recall from Section 2.24, "Analytic Functions":

> **Definition.** If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is a *singular point* of f. That is, z_0 is a singular point if for all $\varepsilon > 0$ there is some $z_1 \in \mathbb{C}$ with $0 < |z_0 - z_1| < \varepsilon$ such that f is analytic at z_1.

Definition. If a function has a singular point z_0 and f is analytic for all z with $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$ then f has an *isolated singularity* at z_0.

Example. If $f(z) = p(z)/q(z)$ is a rational function (that is, p and q are polynomials) then f has isolated singularities at the zeros of q.

Example 2. Consider the principal branch of the logarithm $\text{Log}\, z = \ln r + i\Theta$ where $z = re^{i\Theta}$, $r > 0$, and $-\pi < \Theta < \pi$. Then $\text{Log}\, z$ has a singularity at $z_0 = 0$. Since $\text{Log}\, z$ is not defined (and so not analytic) for z real and negative, then in $0 < |z| < \varepsilon$ there are points where $\text{Log}\, z$ is not analytic; that is. $z_0 = 0$ is not an isolated singularity of $\text{Log}\, z$.

Example 3. Let $\dfrac{1}{\sin(\pi z)}$. then f has a singularity at $z_0 = 0$. Also, f has a singularity for each z where $\sin(\pi/z) = 0$. That is, f has a singularity for all a/n where $n \in \mathbb{Z}$, $n \neq 0$. For $n \in \mathbb{Z}$, $n \neq 0$, these singularities are isolated (just take $\varepsilon = |\pi/n - \pi/(n+1)| = \pi/|n(n+1)|$). However, for $z_0 = 0$ and $\varepsilon > 0$, there are infinitely many singularities of f in $0 < |z| < \varepsilon$ (namely, those at π/n where $n \in \mathbb{Z}$ and $n > \pi/\varepsilon$). So ths singularities at $a_0 = 0$ is not isolated.

Note. If C is a simple closed contour and f is analytic except at a finite number of points, $z_1, z_2, \ldots, z_n$, in C then these points must be isolated singularities of f (take $\varepsilon = \min_{1 \leq i < j \leq n} |z_i - z_j|$).

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