Section 6.69. Residues

Note. In this section we use a Laurent series of a function about an isolated singularity to define the residue of the function at the singularity. The residue is then used to evaluate integrals of the function.

Note. If f has an isolated singularity at z_0 then f is analytic on $0 < |z - z_0| < R_2$ for some $R_2 > 0$ and so by Theorem 60.1, "Laurent's Theorem," f has a Laurent series for $0 < |z - z_0| < R_2$ of

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \text{ where } b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}$$

for $b \in \mathbb{N}$ and C is a positively oriented simple closed contour around z_0 in $0 < |z - z_0| < R_2$. When $n = 1$ this gives

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz \text{ or } \int_C f(z)\,dz = 2\pi i b_1.$$

We'll see that b_1 plays an important role in the evaluation of integrals of f.

Definition. If function f has an isolated singularity at $z = z_0$ then the coefficient b_1 in the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is the *residue* of f at z_0, denoted $b_1 = \text{Res}_{z=z_0} f(z)$.

Note 69.A. We have from above that $\int_C f(z)\,dz = 2\pi i b_1 = 2\pi i \text{Res}_{z=z_0} f(z)$ where C is a positively oriented closed contour around z_0 as described above.

Example 2. Consider $f(z) = \exp(1/z^2)$ and let C be the positively oriented unit circle $|z| = 1$. We now evaluate $\int_C f(z)\,dz$ using residues. Now f is analytic except at $z_0 = 0$, so f has an isolated singularity at $z_0 = 0$ and hence f has a Laurent series centered at $z_0 = 0$ by Theorem 60.1 "Laurents Theorem." We can find the Laurent series by replacing z with $1/z^2$ in the series for $\exp(z)$; that Laurent series will therefore be valid for $0 < |z| < \infty$. We have

$$\exp\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!} = 1 + \frac{0}{z} + \frac{1}{z!z^2} + \frac{0}{z^3} + \frac{1}{2!z^4} + \cdots$$

and so $\text{Res}_{z=0} f(z) = b_1 = 0$. Therefore,

$$\int_C f(z)\,dz = 2\pi i \text{Res}_{z=0} f(z) = 2\pi i b_1 = 0.$$

Brown and Churchill point out that we know that if f is analytic in and on C then $\int_C f(z)\,dz = 0$, but this example shows that the converse does not hold since $\int_C \exp(1/z^2)\,dz = 0$, but $\exp(1/z^2)$ is not analytic inside C.

Example 3. We can use residues and geometric series to evaluate integrals of certain rational functions. Consider $\displaystyle\int_C \frac{dz}{z(z-1)^4}$ where C is the positively oriented circle $|z-2| = 1$. With $f(z) = \dfrac{1}{z(z-2)^4}$, f is analytic on $0 < |z-2| < 2$ and so f has a Laurent series about $z_0 = 2$ by Theorem 60.1, "Laurent's Theorem." Therefore, by Note 69.A, $\displaystyle\int_C \frac{dz}{z(z-2)^4} = 2\pi i \text{Res}_{z=2} f(z)$. Since

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1$$

(the sum of a geometric series), then

$$\frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4}\frac{1}{2+(z-2)} = \frac{1}{2}\frac{1}{(z-2)^4}\frac{1}{1-\left(-\frac{z-2}{2}\right)}$$

$$= \frac{1}{2}\frac{1}{(z-2)^4}\sum_{n=0}^{\infty}\left(\frac{-(z-2)}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-4}.$$

So with $n = 3$, we have the term $\frac{-1}{16}(z-2)^{-1}$ and so $\mathrm{Res}_{z=2}f(z) = \frac{-1}{16}$. So

$$\int_C \frac{dz}{z(z-2)^4} = 2\pi i\left(\frac{-1}{16}\right) = \frac{-\pi i}{8}.$$

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