Section 6.70. Cauchy's Residue Theorem

Note. In this section we extend the use of residues to evaluate integrals from a single isolated singularity to several (but finitely many) isolated singularities.

Note. At the end of Section 68, "Isolated Singular Points," we observed that for simple closed contour C containing singular points $z_1, z_2, \ldots, z_n$ of f, these singular points are isolated, as can be seen by taking $\varepsilon = \min_{1 \le i < j \le n} |z_i - z_j|$. So we can find the residue of f at each such z_k, $\text{Res}_{z=z_k} f(z)$. Then $\int_C f(z)\,dz$ can be calculated in terms of these residues as follows.

Theorem 6.70.1. Cauchy's Residue Theorem.
Let C be as simple closed contour described in the positive sense. If function f is analytic inside and on C except for a finite number of singular points z_k for $k = 1, 2, \ldots, n$ inside C then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z).$$

Example. Consider $\displaystyle\int_C \frac{5z-2}{z(z-1)}\,dz$ where C is the positively oriented circle $|z| = 2$. To use Cauchy's Residue Theorem (Theorem 6.70.1), we must find two residues which means that we must find two Laurent series for f. The isolated singularities of f are at $z = 0$ and $z = 1$. Since

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1$$

(geometric series with ratio z), then

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\frac{-1}{1-z} = -\left(5 - \frac{2}{z}\right)\sum_{n=0}^{\infty} z^n$$

$$= -5\left(\sum_{n=0}^{\infty} z^n\right) + 2\sum_{n=0}^{\infty} z^{n-1} \text{ for } |z| < 1$$

(using the absolute convergence of the series for $|z| < 1$ to rearrange the theorem). With $n = 0$ in the second series, we see that $\text{Res}_{z=0} f(z) = 2$. Next,

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1}\frac{1}{1+(z-1)} = \left(5 + \frac{3}{z-1}\right)\sum_{n=0}^{\infty}(z-1)^n$$

$$= 5\left(\sum_{n=0}^{\infty}(z-1)^n\right) + 3\left(\sum_{n=0}^{\infty}(z-1)^{n-1}\right) \text{ for } |z-1| < 1.$$

With $n = 0$ in the second series, we see that $\text{Res}_{z=1} f(z) = 2$. So by Cauchy's Residue Theorem (Theorem 6.70.1), we see that

$$\int_C \frac{5z-2}{z(z-1)}\, dz = 2\pi i\left(\text{Res}_{z=0} f(z) + \text{Res}_{z=1} f(z)\right) = 2\pi i(2+3) = 10\pi i.$$

Notice that Brown and Churchill observe that this result can also be done, as you would do it in Calculus 2, using partial fractions since the partial fractions (in this case) immediately give to Laurent series.

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