Section 6.71. Residues at Infinity

Note. We introduce a new kind of residue that will allow us to evaluate certain integrals from a single residue, instead of from several residues as we did in the previous section.

Definition. If for function f there is $R_1 > 0$ such that f is analytic for $R_1 < |z| < \infty$ then f has an *isolated singularity point at* $z_0 = \infty$.

Definition. Let f have an isolated singularity point at $z_0 = \infty$, so that f is analytic for $R_1 < |z| < \infty$ for some $R_1 > 0$. Let $R_0 > R_1$ and let C_0 denote the circle $|z| = R_0$ with a negative orientation. The *residue of f at infinity* is

$$\text{Res}_{z=\infty} f(z) = \frac{1}{2\pi i} \int_{C_0} f(z)\, dz.$$

Note. The residue of f at infinity is defined in terms of parameter R_0 which could be any positive number "sufficiently large." However, by Corollary 4.49.B, "Principle of Deformation," any integrals of the form $\int_C f(z)\, dz$, where C is a "sufficiently large" circle with negative orientation, will be equal. So the deformation of $\text{Res}_{z=\infty} f(z)$ is unambiguous (that is, it is *well-defined*).

Note. The reason for giving C_0 a negative orientation in the definition of $\text{Res}_{z=\infty} f(z)$ is to give a consistence with the properties of $\text{Res}_{z=z_0} f(z)$ for finite z_0. For finite z_0, we considered positively oriented simple closed contour C and has $\text{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz$ (see Note 69.A). In this case, as we "travel" along C, the singular points are always to our left. By giving C_0 a negative orientation we always have ∞ on our left.

Note. We use $\text{Res}_{z=\infty} f(z)$ in the proof of the following.

Theorem 6.71.1. If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)\,dz = 2\pi i \text{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

Example. Consider $\displaystyle\int_C \frac{5z-1}{z(z-1)}\,dz$ where C is the circle $|z| = 2$ positively oriented. We worked this example in the previous section by considering two Laurent series for the integrand f. We can now use Theorem 6.71.1 and evaluate it from a single Laurent series. We have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\frac{5/z-2}{z^2(1/z)((1/z)-1)} = \frac{5-2z}{z-z^2} = \frac{5-2z}{z}\frac{1}{1-z} = \left(\frac{5}{z}-2\right)\sum_{n=0}^{\infty}$$

$$= 5\left(\sum_{n=0}^{\infty} z^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} z^n\right) \text{ for } 0 < |z| < 1,$$

(using a geometric series with ratio z where $|z| < 1$ and the absolute convergence to rearrange), and so with $n = 0$ in the first series we see that $\text{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right) = 5$. So by Theorem 6.71.1,

$$\int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i \text{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right) = 2\pi i(z) = 10\pi i.$$

Note. Exercise 6.71.5 states:

> Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that
>
> $$\text{Res}_{z=z_1} f(z) + \text{Res}_{z=z_1} f(z) + \cdots + \text{Res}_{z=z_n} f(z) + \text{Res}_{z=\infty} f(z) = 0.$$

So $\text{Res}_{z=z_1} f(z) + \text{Res}_{z=z_1} f(z) + \cdots + \text{Res}_{z=z_n} f(z) = -\text{Res}_{z=\infty} f(z)$. This is why we can evaluate integrals by considering a single residue at infinity instead of the individual residues. In the proof of Theorem 6.71.1, we see that

$$\text{Res}_{z=\infty} f(z) = \text{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right).$$

Note. We have seen that residues are useful in calculating integrals. But at this stage we must find a Laurent series in order to find a residue. We'll have an easier way soon to evaluate residues in certain cases (see Section 73, "Residues at Poles").

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