Section 6.72. The Three Types of Isolated Singular Points

Note. In this section we use the Laurent series of a function centered at an isolated singular point to classify the singular point into one of three categories.

Definition. If for function f there is $R_1 > 0$ such that f is analytic for $R_1 < |z| < \infty$ then f has an *isolated singularity point at* $z_0 = \infty$.

Note. Suppose f has an isolated singular point at z_0; that is, f is analytic for $0 < |z - z_0| < R_2$ but f is not analytic at z_0, then by Theorem 5.60.1, "Laurent's Theorem,"

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \text{ for } 0 < |z - z_0| < R_2.$$

Definition. Let f have an isolated singular point at $z = z_0$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n = \sum_{n=0}^{\infty} a_z(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}$$

for $0 < |z - z_0| < R_2$ (that is, $c_0 = 0$for $n < -m$). Then z_0 is a *pole of order* m for f. If z_0 is a pole of oder 1 then it is a *simple pole* of f.

Definition. Let f have an isolated singular point at $z = z_0$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n = \sum_{n=0}^{\infty} a_z(z - z_0)^n \text{ for } 0 < |z - z_0| < R_2$$

(that is, $c_n = 0$ for $n < 0$). Then z_0 is a *removable singular point* of f.

Note 6.72.A. If f has a removable singular point at z_0 then we can define $g(z)$ on $|z - z_0| < R_2$ as

$$g(z) = \begin{cases} f(z) & \text{for } 0 < |z - z_0| < R_2 \\ z_0 & \text{for } z = z_0. \end{cases}$$

Then $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and so g is analytic on $|z - z_0| < R_2$. Notice that this means $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} g(z) = g(z_0)$. In this way, the singularity of f at $z = z_0$ has been "removed." An example of this is given by the functions $f(z) = (z^2 - 1)/(z - 1)$ and $g(z) = z + 1$.

Definition. Let f have an isolated singular point at $z = z_0$. If z_0 is neither a removable singular point nor a pole then z_0 is an *essential singular point.*

Note. If f has an essential singular point at $z = z_0$ then the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ for $0 < |z - z_0| < R_2$ must have infinitely many nonzero c_n for $n < 0$.

Example. (Exercise 6.72.1(c)) Consider $f(z) = (\sin z)/z$. Since $\sin z = (e^{-z} - e^{-iz})/2$ (see Section 3.34, "Trigonometric Functions") we have

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \text{ for all } z \in \mathbb{C}$$

and so

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \text{ for } 0 < |z| < \infty$$

and so $(\sin z)/z$ has a removable singularity at $z = 0$. Notice

$$\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \right) = \frac{(-1)^0}{(2(0)+1)!} = 1.$$

Example 3. Consider $f(z) = \dfrac{\sinh z}{z^4}$. Since $\sinh z = (e^z - z^{-z})/2$ (see Section 3.35, "Hyperbolic Functions") we have $\sinh z = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ for all $z \in \mathbb{C}$ and so

$$\frac{\sinh z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n-3}}{(2n+1)!}$$

$$= \frac{1}{z^3} + \frac{1}{6z} + \sum_{n=2}^{\infty} \frac{z^{2n-3}}{(2n+1)!} = \frac{1}{z^3} + \frac{1}{6z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+5)!}$$

and so f has a pole of order $m = 3$ at $z_0 = 0$.

Example 5. Consider $f(z) = \exp(1/z)$. Since $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$ for all $z \in \mathbb{C}$ then

$$\exp(1/z) = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{z^n n!} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \text{ for } 0 < |z| < \infty$$

and so f has an essential singularity at $z_0 = 0$.

Note. Brown and Churchill now mention (without a formal statement) "Picard's Theorem." This claims that if f has an essential singularity at $z = z_0$ then for all $\varepsilon > 0$ such that f is analytic on $0 < |z - z_0| < \varepsilon$, function f assumes every value $c \in \mathbb{C}$ an infinite number of times, with one possible exception for the value c (for $f(z) = \exp(1/z)$ the exceptional value is $c = 0$). For details, see Section XII.1, "The Great Picard Theorem in John Conway's *Functions of One Complex Variable I*, 2nd Edition [Springer-Verlag, 1978]. This is the last section of this graduate-level text, reflecting the background needed to prove the result.

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