Section 6.74. Examples

Note. We now illustrate Theorem 6.73.1 with several examples.

Example 6.74.1. Consider $f(z) = \dfrac{z+1}{z^2+9} = \dfrac{z+1}{(z+3i)(z-3i)}$. Then f has isolated singular points at $z = -3i$ and $z = 3i$. For $z = -3i$ we have, in the notation of Theorem 6.73.1 (notice $m = 1$ and $3i$ is a simple pole), $f(z) = \dfrac{(z+1)(z-3i)}{z+3i} = \dfrac{\varphi(z)}{z+3i}$. So

$$\text{Res}_{z=-3i} f(z) = \varphi(-3i) = \frac{(-3i)+1}{(-3i)-3i} = \frac{-3i+1}{-6i} = \frac{3+i}{6}.$$

Similarly, for $z = 3i$, $f(z) = \dfrac{(z+1)/(z+3i)}{z-3i} = \dfrac{\varphi(z)}{z-3i}$ and

$$\text{Res}_{z=3i} f(z) = \varphi(3i) = \frac{(3i)+1}{(3i)+3i} = \frac{3i+1}{6i} = \frac{3-i}{6}.$$

Example 6.74.3. Consider $f(z) = \dfrac{(\log z)^3}{z^2+1}$ where we use the branch of the logarithm $\log z = \ln r + i\theta$, where $r > 0$ and $0 < \theta < 2\pi$. At singular point $z = i$ we have $f(z) = \dfrac{(\log z)^3/(z+i)}{z-i} = \dfrac{\varphi(z)}{z-i}$ (notice that φ is analytic at $z = i$ and $\varphi(i) \neq 0$) and so

$$\text{Res}_{z=i} f(z) = \varphi(i) = \frac{(\log(i))^3}{(i)+i} = \frac{(\pi i/2)^3}{2i} = \frac{-\pi^3}{16}.$$

Similarly, at $z = i$,

$$\text{Res}_{z=-i} f(z) = \frac{(\log(-i))^3}{(-i)-i} = \frac{(3\pi i/2)^3}{-2i} = \frac{27\pi^3}{16}.$$

Example 6.74.5. Consider $f(z) = \dfrac{1}{z(e^z - 1)}$. Then f has an isolated singular point at $z = 0$. Now $e^z = \sum_{n=0}^{\infty} z^n/n!$ so, by the absolute convergence of this series for $|z| < \infty$ (see Section 5.67),

$$z(e^z - 1) = z\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) = z\left(\sum_{n=1}^{\infty} \frac{z^n}{z!}\right) = z^2\left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}\right) = z^2\left(\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}\right).$$

So

$$f(z) = \frac{1}{z(e^z - 1)} = \frac{1}{z^2}\left(\frac{1}{\sum_{n=0}^{\infty} z^n/(n+1)!}\right) = \frac{\varphi(z)}{z^2}$$

where $\varphi(z) = 1\,/\sum_{n=0}^{\infty} z^n/(n+1)!$. So φ is analytic and nonzero at $z = 0$. Therefore, by Theorem 6.73.1, f has a pole of order 2 at $z = 0$ and

$$\mathrm{Res}_{z=0} f(z) = \frac{\varphi^{(2-1)}(0)}{(2-1)!} = \varphi'(0) = \left.\frac{-\sum_{n=1}^{\infty} nz^{n-1}/(n+1)!}{\left(\sum_{n=0}^{\infty} z^n/(n+1)!\right)^2}\right|_{z=0} = \frac{-1/2}{(1)^2} = \frac{-1}{2}.$$

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