Section 6.75. Zeros of Analytic Functions

Note. In this section we give the definition of a zero of "order" m of an analytic function and show (in Theorem 6.75.1) that it is similar to a zero of a polynomial of multiplicity of m.

Definition. Suppose f is analytic at z_0. If $f(z_0) = 0$ and if there is $m \in \mathbb{N}$ such that $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$, then f has a *zero of order* m at z_0.

Note. We now show that an analytic function can be factored much in the same way that the Factor Theorem allows us to factor polynomials.

Theorem 6.75.1. Let function f be analytic at z_0. It has a zero of order m at z_0 if and only if there is a function g which is analytic and nonzero at z_0 such that $f(z) = (z - z_0)^m g(z)$.

Example 6.75.2. Consider the entire function $f(z) = z(e^z - 1)$ (which we also encountered in Example 6.74.5). Notice $f(0) = 0$, $f'(z) = [1](e^z - 1) + (z)[e^z] = ze^z + e^z - 1$, $f'(0) = 0$, $v''(z) = [1](e^z) + (z)[e^z] + e^z = ze^z + 2e^z$, and $f''(0) = 2$. So f has a zero of order $m = 2$ at $z_0 = 0$. Hence by Theorem 6.75.1, $f(z) = z^2 g(z)$

for entire g where $g(0) \neq 0$. In fact,

$$g(z) = \begin{cases} (e^z - 1)/z & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

and $g(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$ (see example 6.74.5).

Note. The previous theorem associates an order with a zero of an analytic function. The next result shows that the zeros of a nonzero analytic function are isolated.

Theorem 6.75.2. Given a function f and a point z_0, suppose that

(a) f is analytic at z_0,

(b) $f(z_0) = 0$ but f is not identically equal to zero in any neighborhood of z_0.

Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

Note. The following is somewhat of a converse of Theorem 6.75.2.

Theorem 6.75.3. Let f be a function and let z_0 a point where

(a) f is analytic throughout a neighborhood N_0 of z_0 and with power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $z \in N_0$, and

(b) $f(z) = 0$ at each point z of a domain D or a line segment L containing z_0.

Then $f(z) \equiv 0$ in N_0. That is, f is identically equal to zero throughout N_0.