Section 6.76. Zeros and Poles

Note. We now relate the zeros of order m for an analytic function to the poles of order m in the reciprocal of the analytic function.

Theorem 6.76.1. Suppose that

(a) two functions p and q are analytic at a point z_0, and

(b) $p(z_0) \neq 0$ and q has a zero of order m at z_0.

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

Example 6.76.1. Let $p(z) = 1$ and $z(z) = z(e^z - 1)$. Then p and q are entire and by Example 6.75.2, q has a zero of order $m = 2$ at $z_0 = 0$. So by Theorem 6.76.a, $\dfrac{p(z)}{q(z)} = \dfrac{1}{z(e^z - 1)}$ has a pole of order 2 at $z_0 = 0$ (as we saw in Example 6.74.5).

Theorem 6.76.2. Let the functions p and q be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) = 0$ (that is, q has a zero of multiplicity one at z_0) then z_0 is a simple pole of p/q and $\text{Res}_{z=z_0} \dfrac{p(z)}{q(z)} = \dfrac{p(z_0)}{q'(z_0)}$.

Example 6.76.2. Consider $f(z) = \cos z / \sin z = \cot z$. With $p(z) = \cos z$ and $q(z) = \sin z$. Since $a(z) = \sin z = 0$ for $z = n\pi$ where $n \in \mathbb{Z}$, $q'(z) = \cos z$, $q'(n\pi) = (-1)^n \neq 0$, and $p(n\pi) = (-1)^n \neq 0$ then by Theorem 6.76.2, f has a simple pole at each $n\pi$ where $n \in \mathbb{Z}$ and

$$\text{Res}_{z=n\pi} f(z) = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.$$

Example 6.76.4. Consider $f(z) = \dfrac{z}{z^4+4}$. Let $p(z) = z$ and $q(z) = z^4 + 4$. Then for $z_0 = \sqrt{2}\exp(i\pi/4) = 1 + i$ and $p(z_0) = z_0 \neq 0$, $q(z_0) = 0$, and $q'(z_0) = 4z_0^3 \neq 0$. So by Theorem 6.76.2, f has a simple pole at z_0, and the residue at z_0 is

$$\text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{4(2\exp(i\pi/2))} = \frac{1}{8i} = \frac{-i}{8}.$$

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